Article

On some Theoretical Considerations of Fluids in Motion and their Internal Friction

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List of Symbols

a	Radius of sphere, disc, surface, etc.
b	Radius of sphere, disc, surface, etc.
C, C_0	Velocity
e', e'', e'''	Roots of Equation 2.4, s^{-1}
$\mathscr{F}$	Force
g	force of gravity
G	moment of force of fluid
h	distance between two points depending on context
$\mathfrak{i}$	imaginary unit $= \sqrt{-1}$
k	cubical contraction or factor arising from taking into account the increase of inertia by that of a mass equal to k times the mass of the fluid displaced
k, k'	coefficients according to Equation 2.52
l	Length of wire of pendulum, positive number
$l, m, n, l', m', n', l'', \ldots$	direction-cosines of angles
m	ratio or positive number, depending on context, unit m^{-2}
$\mathfrak{m}$	modulus of ma, see Equation 2.100
M, M'	(Virtual) Mass of sphere, disc, etc., point of mass, mass displaced
n	positive number, depending on context, unit s^{-1}
$\mathfrak{n}$	factor n of Baily's paper by which the correction for buoyancy must be multiplied
$Ox_\prime, Oy_\prime, Oz_\prime$	rectangular axes, parallel to the axes of extension, see page 77
p	pressure, $\mathrm{kg\,m^{-1}\,s^{-2}}$
r	Radius vector
t	Time (of oscillation, etc.)

$t_\prime$	tangential force or tangential action respectively
T_3	Tangential pressure in the direction of y
U, V, W	components of the velocity of translation
u	component of the velocity along the rectangular axes of x
v	component of the velocity along the rectangular axes of y
w	component of the velocity along the rectangular axes of y
x, y, z	Cartesian coordinates
x', y', z'	Relative Cartesian coordinates to x, y, z
X, Y, Z	Components of the accelerating force or force acting on fluid respectively
α	angle, (increments of) co-ordinate
β	angle, (increments of) co-ordinate
γ	angle, radius or distance of sphere, disc, etc. to the axis of rotation or distance from the pivot point respectively, (increments of) co-ordinate
θ	angle
Λ	$= -\left(\ln(4) + \frac{1}{\sqrt{\pi}} \cdot \Gamma'\left(\frac{1}{2}\right)\right)$, see Equation 2.102.
μ	dynamic viscosity (in modern notation, unit $\mathrm{kg\,m^{-1}\,s^{-1}}$), moment of inertia, unit $\mathrm{kg\,s^2}$, mass of the fluid displaced
μ'	index of friction (or kinematic viscosity in modern notation, unit $\mathrm{m^2\,s^{-1}}$) $\mu' = \frac{\mu}{\rho}$, moment of inertia, unit $\mathrm{kg\,s^2}$
ν	coefficient according to Equation 2.42 (with $n = \frac{\pi}{\tau}$)
Π	pressure
ϖ_1	$= \frac{\mathrm{d}u}{\mathrm{d}t} + u\frac{\mathrm{d}u}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}y} + w\frac{\mathrm{d}u}{\mathrm{d}z}$
ϖ_2	$= \frac{\mathrm{d}u}{\mathrm{d}t} + u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}v}{\mathrm{d}y} + w\frac{\mathrm{d}v}{\mathrm{d}z}$
ϖ_3	$= \frac{\mathrm{d}u}{\mathrm{d}t} + u\frac{\mathrm{d}w}{\mathrm{d}x} + v\frac{\mathrm{d}w}{\mathrm{d}y} + w\frac{\mathrm{d}w}{\mathrm{d}z}$
ρ	density of surrounding fluid, density of water
σ	density of air or sphere, velocity of the point (x, y, z)
τ	time of oscillation, time interval

$\mathrm{d}\phi$	exact differential, $= u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$
$\mathrm{d}\chi$	exact differential, $= u\,\mathrm{d}y - v\,\mathrm{d}x$, see Equation 2.68
$\mathrm{d}\psi$	exact differential, $= \varpi(u\,\mathrm{d}\varpi - q\,\mathrm{d}x)$, see Equation 2.20
$\omega, \omega' \ldots$	angle, angular velocity

Part I.

On some Cases of Fluid Motion

1. Introductory Notes

The equations of Hydrostatics are founded on the principles that the mutual action of two adjacent elements of a fluid is normal to the surface which separates them, and that the pressure is equal in all directions. The latter of these is a necessary consequence of the former, as has been shewn by Mr. Airy[1]. An exactly similar proof may be employed in Hydrodynamics, by which it may be shewn that, if the mutual action of two adjacent elements of a fluid in motion is normal to their common surface, the pressure must be equal in all directions, in order that the accelerating force which acts on the centre of gravity of an element may not become infinite, when we suppose the dimensions of the element indefinitely diminished. In Hydrostatics, the accurate agreement of the results of our calculations with experiments, (those phenomena which depend on capillary attraction being excepted), fully justifies our fundamental assumption. The same assumption is made in Hydrodynamics, and from it are deduced the fundamental equations of fluid motion. But the verification of our fundamental law in the case of a fluid at rest, does not at all prove it to be true in the case of a fluid in motion, except in the very limited case of a fluid moving as if it were solid. Thus, oil is sufficiently fluid to obey the laws of fluid equilibrium, (at least to a great extent), yet no one would suppose that oil in motion ought to be considered a perfect fluid. It would appear from the following consideration, that the fluidity of water and other such fluids is not quite perfect. When a mass of water contained in a vessel of the form of a solid of revolution is stirred round, and then left to itself, it presently comes to rest. This, no doubt, is owing to the friction against the sides of the vessel. But if the fluidity of water were perfect[2], it does not appear how the retardation due to this friction could be transmitted through the mass. It would appear that in that case a thin film of fluid close to the sides of the vessel would remain at rest,[3] the remaining part of the fluid being unaffected by it. And in this respect, that part of Poisson's solution of the problem of an oscillating sphere, which relates to friction, appears to me in some degree unsatisfactory. A term enters into the equation of motion of the sphere depending on the friction of the fluid on the sphere, while no such term enters into the equations of motion of the fluid, to express the equal and opposite friction of the sphere on the fluid. In fact, as long as we regard the fluidity of the fluid as perfect, no such term can enter. The only way by which to estimate the extent to which the imperfect fluidity of

[1]See also Professor Miller's *Hydrostatics*, page 2.

[2]This is the case for liquid helium-4, a Superfluid which behaves like a fluid with zero viscosity.

[3]In nowadays language a *Boundary Layer*

fluids may modify the laws of their motion, without making any hypothesis as to the molecular constitution of fluids, appears to be, to calculate according to the hypothesis of perfect fluidity some cases of fluid motion, which are of such a nature as to be capable of being accurately compared with experiment. The cases of that nature which have hitherto been calculated, are by no means numerous. My object in the present paper which I have the honour to lay before the Society, has been partly to calculate some such cases which may be useful in determining how far we are justified in regarding fluids as perfectly fluid, and partly to give examples of the methods by which the solution of problems depending on partial differential equations may be effected.

In the first seven articles, I have mentioned and explained some general principles, which are afterwards applied. Some of these are not new, but it was convenient to state them for the sake of reference. Others are I believe new, at least in their development. In the remaining articles, I have given different problems, of which I have succeeded in obtaining the solutions. As the problem to be solved is usually stated at the head of each article, I shall here only mention some of the results. As a particular case of the problem given in Art. 8., page 29, I find that, when a cylinder oscillates in an infinitely extended fluid, the effect of the inertia of the fluid is to increase the mass of the cylinder by that of the fluid displaced. In part of Art. 9., page 36, I find that when a ball pendulum oscillates in a concentric spherical envelope, the effect of the inertia of the fluid is to increase the mass of the ball by $\frac{b^3+2a^3}{2\cdot(b^3-a^3)}$ times that of the fluid displaced, a being the radius of the ball, and b that of the envelope. Poisson, in his solution of the problem of the sphere, arrives at the strange result that the envelope does not at all retard the oscillating sphere. I have pointed out the erroneous step by which he was led to this conclusion, which I am clearly called upon to do, in venturing to differ from so high an authority. Of the different cases of fluid motion which I have given, that which appears to be capable of the most accurate and varied comparison with experiment, is the motion of fluid in a rectangular box which is closed on all sides, given in Art. 13., page 57. The experiment consists in comparing the calculated and observed times of oscillation. I find that when the motion is small, the effect of the fluid on the motion of the box is the same as that of a solid having the same mass, centre of gravity, and principal axes, but having different moments of inertia, these moments being given by infinite series, which converge with great rapidity. I have also in Art. 11., page 47, given some cases of progressive motion, deduced on the supposition that the same particles of fluid remain in contact with the solid, which do not at all agree with experiment.

In almost all the cases given in this paper, the problem of finding the permanent state of temperature in the several solids considered, supposing the surfaces of those solids kept up to constant temperatures varying from point to point, may be solved by a similar analysis. I find that some of these cases have been already solved by C. Duhamel in a paper "*Sur la méthode générale relative au mouvement de la chaleur dans les corps solides plongés dans des milieux dont la température varie*

avec le temps" inserted in the 22nd *Cahier* of the *Journal de l'École Polytechnique. Paris*, **14**, 1833.[4] The cases alluded to are those of the temperature in a solid sphere, and in a rectangular parallelepiped. Since, however, the application of the formulæ in the two cases of fluid motion and of the permanent state of temperature is different, as well as the formulæ themselves to a certain extent, I thought it might be worth while to give them.

[4] `https://gallica.bnf.fr/ark:/12148/cb34378280v/date`

2. Theoretical Considerations on the Motion of Incompressible Elastic Fluids

1. The investigations in this paper apply directly to incompressible fluids, as the fluids spoken of will be supposed to be, unless the contrary is stated. The motions of elastic fluids may in most cases be divided into two classes, one consisting of those condensations on which sound depends, the other, of those motions which the fluid takes in consequence of the motion of solid bodies in it. Those motions of the fluid, which take place in consequence of very rapid motions of solids, (such as those of bullets), form a connecting link between these two classes. The motions of the second class are, it is true, accompanied by condensations, and propagated with the velocity of sound, but if the motions of the solids are not great we may, without sensible error, suppose the motions of the fluid propagated instantaneously to distances where they cease to be sensible, and may neglect the condensation. The investigations in this paper will apply without sensible error to this kind of motion of elastic fluids.

In all cases also the motion will be supposed to begin from rest, which allows us to suppose that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential $\mathrm{d}\phi$ where u, v and w are the components, parallel to the axes of x, y, and z, of the whole velocity of any particle. In applying our investigations however to fluids such as they exist in nature, this principle must not be strained too far. When a body is made to revolve continually in a fluid, the parts of the fluid near the body will soon acquire a rotatory motion, in consequence, in all probability, of the mutual friction of the parts of the fluid; so that after a time $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ could no longer be taken an exact differential. It is true that in motion in two dimensions there is one sort of rotatory motion for which that quantity is an exact differential; but if a closed vessel, filled with fluid at first at rest, be made to revolve uniformly round a fixed axis, the fluid will soon do so too, and therefore that quantity will cease to be an exact differential. For the same reason, in the progressive motion of a solid in a fluid, the effect of friction continually accumulating, the motion might at last be sensibly different from what it would be if there were no friction, and that, even if the friction were very small. In the case of small oscillatory motions however it would appear that the effect of friction in the forward oscillation, supposing that friction small, would be counteracted by its effect in the backward oscillation, at

least if the two were symmetrical. In this case then we might expect our results to agree very nearly with experiment, so far at least as the *time* of oscillation is concerned.
The forces which act on the fluid are supposed in the following investigations to be such that $X\,\mathrm{d}x + Y\,\mathrm{d}y + Z\,\mathrm{d}z$ is the exact differential of a function of x, y and z, where X, Y, Z, are the components, parallel to the axes, of the accelerating force acting on the particle whose co-ordinates are x, y, z. The only effect of such forces, in the case of a homogeneous, incompressible fluid, being to add the quantity $\rho \cdot \int (X\,\mathrm{d}x + Y\,\mathrm{d}y + Z\,\mathrm{d}z)$ to the pressure, the forces, as well as the pressure due to them, will for the future be omitted for the sake of simplicity.

2. It is a recognised principle, and one of great importance in these investigations, that when a problem is determinate, any solution which satisfies all the requisite conditions, no matter how obtained, is the solution of the problem. In the case of fluid motion, when the initial circumstances and the conditions with respect to the boundaries of the fluid are given, the problem is determinate. If it were required to find what sort of steady motion could take place between given surfaces, the problem would not be determinate, since different kinds of steady motion might result from different initial circumstances.
It may be well here to enumerate the conditions which must be satisfied in the case of a homogeneous incompressible fluid without a free surface, the case which is considered in this paper. We have first the equations,

$$\left.\begin{aligned} \frac{1}{\rho}\cdot\frac{\mathrm{d}p}{\mathrm{d}x} &= -\varpi_1\,, \\ \frac{1}{\rho}\cdot\frac{\mathrm{d}p}{\mathrm{d}y} &= -\varpi_2\,, \\ \frac{1}{\rho}\cdot\frac{\mathrm{d}p}{\mathrm{d}z} &= -\varpi_3\,, \end{aligned}\right\} \qquad \text{(A)}$$

putting

$$\varpi_1 = \frac{\mathrm{d}u}{\mathrm{d}t} + u\frac{\mathrm{d}u}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}y} + w\frac{\mathrm{d}u}{\mathrm{d}z}$$

and ϖ_2, ϖ_3 for the corresponding quantities for y and z, and omitting the forces. We have also the equation of continuity,

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z} = 0\,. \qquad \text{(B)}$$

Equation A and Equation B hold at all times for all points of the fluid mass.
If σ be the velocity of the point (x, y, z) of the surface of a solid in contact with the fluid resolved along the normal, and ν the velocity, resolved along the same normal, of the fluid particle, which at the time t is in contact with the above point

of the solid, we must have[1]

$$\nu = \sigma \tag{a}$$

at all times and for all points of the fluid which are in contact with a solid.
If the fluid extend to infinity, and the motion at first be zero at an infinite distance, we must have

$$u = 0\,, \quad v = 0\,, \quad w = 0\,, \quad \text{at an infinite distance.} \tag{b}$$

An analogous condition is, that the motion shall not become infinitely great about a particular point, as the origin.
Lastly, if u_0, v_0, w_0, be the initial velocities, subject of course to satisfy Equation B and Equation a, we must have

$$u = u_0\,, \quad v = v_0\,, \quad w = w_0\,, \quad \text{when} \;\; t = 0\,. \tag{c}$$

In the most general cases the equations which u, v and w are to satisfy at every point of the mass and at every time are Equation B and the three equations

$$\left.\begin{aligned} \frac{\mathrm{d}\varpi_1}{\mathrm{d}y} &= \frac{\mathrm{d}\varpi_2}{\mathrm{d}x}\,, \\ \frac{\mathrm{d}\varpi_2}{\mathrm{d}z} &= \frac{\mathrm{d}\varpi_3}{\mathrm{d}y}\,, \\ \frac{\mathrm{d}\varpi_3}{\mathrm{d}x} &= \frac{\mathrm{d}\varpi_1}{\mathrm{d}z}\,. \end{aligned}\right\} \tag{C}$$

These equations being satisfied, the quantity $\varpi_1\,\mathrm{d}x + \varpi_2\,\mathrm{d}y + \varpi_3\,\mathrm{d}z$ will be an exact differential, whence p may be determined by integrating the value of $\mathrm{d}p$ given by Equation A. Thus the condition that these latter equations shall be satisfied is equivalent to the condition that the Equation C shall be satisfied.
In nearly all the cases considered in this paper, and in all those of which the complete solution is given, the motion is such that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential $\mathrm{d}\phi$. This being the case, the Equation C are, as it is well-known, always satisfied, the value of p being given by the equation

$$\frac{p}{\rho} = \psi(t) - \frac{\mathrm{d}\phi}{\mathrm{d}t} - \frac{1}{2}\cdot\left\{\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}\phi}{\mathrm{d}y}\right)^2 + \left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2\right\}\,, \tag{D}$$

$\psi(t)$ being an arbitrary function of t, which may if we please be included in ϕ. In this case, therefore, the *single* condition which has to be satisfied at all times, and

[1]For greater clearness, those equations which must hold for all values of the variables, within limits depending on the problem are denoted by capitals, while those which hold only for certain values of the variables, or of some of them, are denoted by small letters. The latter class serve to determine the forms of the arbitrary functions contained in the integrals of the former.

at every point of the mass is Equation B, which becomes in this case

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} + \frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} = 0\,. \tag{E}$$

In the case of impulsive motion, if u_0, v_0, w_0, be the velocities just before impact, u, v, w, the velocities just after, and q the impulsive pressure, the Equation A are replaced by the equations

$$\left.\begin{aligned} \frac{1}{\rho}\cdot\frac{\mathrm{d}q}{\mathrm{d}x} &= -u + u_0\,, \\ \frac{1}{\rho}\cdot\frac{\mathrm{d}q}{\mathrm{d}y} &= -v + v_0\,, \\ \frac{1}{\rho}\cdot\frac{\mathrm{d}q}{\mathrm{d}z} &= -w + w_0\,,\,, \end{aligned}\right\} \tag{F}$$

and in order that these equations may be satisfied it is necessary and sufficient that $(u-u_0)\,\mathrm{d}x + (v-v_0)\,\mathrm{d}y + (w-w_0)\,\mathrm{d}z$ be an exact differential $\mathrm{d}\phi$, which gives

$$q = C - \rho\cdot\phi\,.$$

The only equation which must be satisfied at every point of the mass is Equation B, which is equivalent to Equation E, since by hypothesis u_0, v_0, and w_0 satisfy Equation B. The conditions according to Equation a and Equation b remain the same as before.

One observation however is necessary here. The values of u, v and w are always supposed to alter continuously from one point in the interior of a fluid mass to another. At the extreme boundaries of the fluid they may however alter abruptly. Suppose now values of u, v and w to have been assigned, which do not alter abruptly, which satisfy Equation B and Equation C as well as the conditions according to Equation a, Equation b and Equation c, or, to take a particular case, values which do not alter abruptly, which satisfy the Equation B and the same conditions, and which render $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ an exact differential. Then the values of $\frac{\mathrm{d}p}{\mathrm{d}x}$, $\frac{\mathrm{d}p}{\mathrm{d}y}$ and $\frac{\mathrm{d}p}{\mathrm{d}z}$ will alter continuously from one point to another, but it does not follow that the value of p itself cannot alter abruptly. Similarly in impulsive motion the value of q may alter abruptly, although those of $\frac{\mathrm{d}q}{\mathrm{d}x}$, $\frac{\mathrm{d}q}{\mathrm{d}y}$ and $\frac{\mathrm{d}q}{\mathrm{d}z}$ alter continuously. Such abrupt alterations are, however, inadmissible; whence it follows as an additional condition to be satisfied,

> that the value of p or q, obtained by integrating Equation A or Equation F, shall not alter abruptly from one point of the fluid to another. (d)

An example will make this clearer. Suppose a mass of fluid to be at rest in a finite cylinder, whose axis coincides with that of z, the cylinder being entirely filled, and closed at both ends. Suppose the cylinder to be moved by impact with an initial velocity C in the direction of x; then shall

$$u = C\,, \quad v = 0\,, \quad w = 0\,.$$

For these values render $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ an exact differential $\mathrm{d}\phi$, where ϕ satisfies Equation E; they also satisfy Equation a; and, lastly, the value of q obtained by integrating Equation F, namely, $C' - C\rho x$, does not alter abruptly. But if we had supposed that ϕ was equal to $Cx + C'\theta$, where $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, the Equation E and the condition according to Equation a would still be satisfied, but the value of q would be $C'' - \rho \cdot (Cx + C'\theta)$, in which the term $\rho C'\theta$ alters abruptly from $2\pi\rho C'$ to 0, as θ passes through the value 2π. The condition according to Equation d then alone shews that the former and not the latter is the true solution of the problem.

The fact that the analytical conditions of a problem in fluid motion, as far as those conditions depend on the velocities, may be satisfied by values of those velocities, which notwithstanding correspond to a pressure which alters abruptly, may be thus explained. Conceive two masses of the same fluid contained in two similar and equal closed vessels A and B. For more simplicity, suppose these vessels and the fluid in them to be at first at rest. Conceive the fluid in B to be divided by an infinitely thin lamina which is capable of assuming any form, and, at the same time, of sustaining pressure. Suppose the vessels A and B to be moved in exactly the same manner, the lamina in B being also moved in any arbitrary manner. It is clear that, except for one particular motion of the lamina, the motion of the fluid in B will be different from that of the fluid in A. The velocities u, v, w, will in general be different on opposite sides of the lamina in B. For particular motions of the lamina however the velocities u, v, w, may be the same on opposite sides of it, while the pressures are different. The motion which takes place in B in this case might, only for the condition according to Equation d, be supposed to take place in A.

It is true that Equation A or Equation F, could not strictly speaking be said to hold good at those surfaces where such a discontinuity should exist. Still, to avoid the liability to error, it is well to state the condition according to Equation d distinctly. When the motion begins from rest, not only must $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ be an exact differential $\mathrm{d}\phi$, and u, v, w, not alter abruptly, but also ϕ must not alter abruptly, provided the particles in contact with the several surfaces remain in contact with those surfaces; for if this condition be not fulfilled, the surface for which it is not fulfilled will as it were cut the fluid into two. For it follows from the Equation D that $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ must not alter abruptly, since otherwise p would alter abruptly from one point of the fluid to another; and $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ neither altering abruptly nor becoming infinite, it follows that ϕ will not alter abruptly. Should an impact occur at any period of the motion, it follows from Equation F that this cannot cause the value of ϕ to alter abruptly, since such an abrupt alteration would give a corresponding abrupt

alteration in the value of q.

3. A result which follows at once from the principle laid down in the beginning of the last article is this, that when the motion of a fluid in a closed vessel which is at rest, and is completely filled, is of such a kind that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, it will be steady. For let u, v, w, be the initial velocities, and let us see if the velocity at the same point can remain u, v, w. First, $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ being an exact differential, Equation A will be satisfied by a suitable value of p, which value is given by Equation D. Also Equation B is satisfied since it is so at first. The condition according to Equation a becomes $\nu = 0$, which is also satisfied since it is satisfied at first. Also the value of p given by Equation D will not alter abruptly, for $\frac{\mathrm{d}\phi}{\mathrm{d}t} = 0$, or a function of t, and the velocities $\frac{\mathrm{d}\phi}{\mathrm{d}x}$ &c. are supposed not to alter abruptly. Hence, all the requisite conditions are satisfied; and hence, (Art. 2., page 18) the hypothesis of steady motion is correct.[2]

4. In the case of an incompressible fluid, either of infinite extent, or confined, or interrupted in any manner by any solid bodies, if the motion begins from rest, and if there be none of the cutting motion mentioned in Art. 2., page 18, the motion at the time t will be the same as if it were produced instantaneously by the impulsive motion of the several surfaces which bound the fluid, including among these surfaces those of any solids which may be immersed in it. For let u, v, w, be the velocities at the time t. Then by a known theorem $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ will be an exact differential $\mathrm{d}\phi$, and ϕ will not alter abruptly (Art. 2., page 18). ϕ must also satisfy Equation E, and the conditions according to Equation a and Equation b. Now if u', v', w', be the velocities on the supposition of an impact, these quantities must be determined by precisely the same conditions as u, v, and w. But the problem of finding u', v', w', being evidently determinate, it follows that the identical problem of finding u, v and w is also determinate, and therefore the two problems have the same solution; so that

$$u = u'\,, \quad v = v'\,, \quad w = w'\,.$$

This principle has been mentioned by M. Cauchy, in a memoir entitled *Mémoire sur la theorie de la propagation des ondes a la surface d'un fluide pesant*, in the first volume of the Mémoires presented to the French Institute, page 14, 1827. It will be employed in this paper to simplify the requisite calculations *by enabling us to dispense with all consideration of the previous motion*, in finding the motion of the fluid at any time in terms of that of the bounding surfaces. One simple deduction from it is that, when all the bounding surfaces come to rest, each element of the

[2]N.B. It is only within a space which is at least doubly connected that such a motion is possible. Thus in the example given in the preceding article , the axis of the cylinder, where the velocity becomes infinite, may be regarded as an infinitely slender core which we are forbidden to cross, and which renders the space within the cylinder virtually ring-shaped.

fluid will come to rest. Another is, that if the velocities of the bounding surfaces are altered in any ratio, the value of ϕ will be altered in the same ratio.

3. Superposition of different motions

5. In calculating the initial motion of a fluid, corresponding to given initial motions of the bounding surfaces, we may resolve the latter into any number of systems of motions, which when compounded give to each point of each bounding surface a velocity, which when resolved along the normal is equal to the given velocity resolved along the same normal, provided that, if the fluid be enclosed on all sides, each system be such as not to alter its volume. For let u', v', w', ν', σ', be the values of u, v, &c., corresponding to the first system of motions; u'', v'', &c., the values of those quantities corresponding to the second system, and so on; so that

$$\begin{aligned} u &= u' + u'' + \cdots , \\ v &= v' + v'' + \cdots , \\ w &= w' + w'' + \cdots , \\ \nu &= \nu' + \nu'' + \cdots , \\ \sigma &= \sigma' + \sigma'' + \cdots . \end{aligned}$$

Then since we have by hypothesis $u'\,\mathrm{d}x + v'\,\mathrm{d}y + w'\,\mathrm{d}z$ an exact differential $\mathrm{d}\phi'$, $u''\,\mathrm{d}x + v''\,\mathrm{d}y + w''\,\mathrm{d}z$ an exact differential $\mathrm{d}\phi''$, and so on, it follows that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential. Again by hypothesis $\nu' = \sigma'$, $\nu'' = \sigma''$, &c., whence $\nu = \sigma$. Also, if the fluid extend to an infinite distance, u, v, and w must there vanish, since that is the case with each of the systems u', v', w', &c. Lastly, the quantities ϕ', ϕ'', &c., not altering abruptly, it follows that ϕ, which is equal to $\phi' + \phi'' + \cdots$, will not alter abruptly. Hence the compounded motion will satisfy all the requisite conditions, and therefore, (Art. 2., page 18) it is the actual motion.
It will be observed that the pressure p will not be obtained by adding together the pressures due to each of the above systems of velocities. To find p we must substitute the complete value of ϕ in Equation D. If, however, the motion be very small, so that the square of the velocity is neglected, it will be sufficient to add together the several pressures just mentioned.
In general the most convenient systems into which to decompose the motion of the bounding surfaces are those formed by considering the motion of each surface, or of a certain portion of each surface, separately. Such a portion may be either finite or infinitesimal. In fact, in some of the cases of motion that will be presently given, where ϕ is expressed by a double integral with a function under the integral sign expressing the motion of the bounding surfaces, it will be found that each element of the integral gives a value of ϕ such that, except about the corresponding element

of the bounding surface, the motion of all particles in contact with those surfaces is tangential.
A result which follows at once from this principle, and which appears to admit of comparison with experiment, is the following. Conceive an ellipsoid, or any body which is symmetrical with respect to three planes at right angles to each other, to be made to oscillate in a fluid in the direction of each of its three axes in succession, the oscillations being very small. Then, in each case, as may be shewn by the same sort of reasoning as that employed in Art. 8., page 29, in the case of a cylinder, the effect of the inertia of the fluid will be to increase the mass of the solid by a mass having a certain unknown ratio to that of the fluid displaced. Let the axes of co-ordinates be parallel to the axes of the solid; let x, y, z, be the co-ordinates of the centre of the solid, and let M, M' M'', be the imaginary masses which we must suppose added to that of the solid when it oscillates in the direction of the axes of x, y, z, respectively. Let it now be made to oscillate in the direction of a line making angles α, β, γ, with the axes, and let s be measured along this line. Then the motions of the fluid due to the motions of the solid in the direction of the three axes will be superimposed. The motion being supposed to be small, the resultant of the pressures of the fluid on the solid will be three forces, equal to

$$M\cos(\alpha)\cdot\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\,,\quad M'\cos(\beta)\cdot\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\,,\quad M''\cos(\gamma)\cdot\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\,,$$

respectively, in the directions of the three axes. The resultant of these in the direction of the motion will be $M_{\prime}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$, where

$$M_{\prime} = M\cdot\cos^2(\alpha) + M'\cdot\cos^2(\beta) + M''\cdot\cos^2(\gamma)\,.$$

Each of the quantities M, M', M'' and $M_{\prime}$, may be determined by observation, and we may find whether the above relation holds between them. Other relations of the same nature may be deduced from the principle explained in this article.

4. Reflection

6. Conceive two solids, A and B, immersed in a fluid of infinite extent, the whole being at rest. Suppose A to be moved in any manner by impulsive forces, while B is held at rest. Suppose the solids A and B of such forms that, if either were removed, and the several points of the surface of the other moved instantaneously in any given manner, the motion of the fluid could be determined: then the actual motion can be approximated to in the following manner. Conceive the place of B to be occupied by fluid, and A to receive its given motion; then by hypothesis the initial motion of the fluid can be determined. Let the velocity with which the fluid in contact with that which is supposed to occupy B's place penetrates into the latter be found, and then suppose that the several points of the surface of B are moved with normal velocities equal and opposite to those just found, A's place being supposed to be occupied by fluid. The motion of the fluid corresponding to the velocities of the several points of the surface of B can then be found, and A must now be treated as B has been, and so on. The system of velocities of the particles of the fluid corresponding to the first system of velocities of the particles of the surface of B, form what may be called *the motion of A reflected from B*; the motion of the fluid arising from the second system of velocities of the particles of the surface of A may be called *the motion of A reflected from B and again from A*, and so on. It must be remembered that all these motions take place simultaneously. It is evident that these reflected motions will rapidly decrease, at least if the distance between A and B is considerable compared with their diameters, or rather with the diameter of either. In this case the calculation of one or two reflections will give the motion of the fluid due to that of A with great accuracy. It is evident that the principle of reflection will extend to any number of solid bodies immersed in a fluid; or again, the body B may be supposed to be hollow, and to contain the fluid and A, or else A to contain B. In some cases the series arising from the successive reflections can be summed, in which case the motion will be determined exactly. The principle explained in this article has been employed in other subjects, and appears likely to be of great use in this. It is the same for instance as that of *successive influences* in Electricity.

7. If a mass of fluid be at rest or in motion in a closed vessel which it entirely fills, the vessel being either at rest or moving in any manner, any additional motion of translation communicated to the vessel will not affect the relative motion of the fluid. For it is evident that on the supposition that the relative motion is not affected the Equation B and the condition according to Equation a will still

be satisfied. Also, if ϖ_1, ϖ_2, ϖ_3, be the components of the effective force of any particle in the first case, and U, V, W, be the components of the velocity of translation, then

$$\varpi_1 + \frac{\mathrm{d}U}{\mathrm{d}t}\,, \quad \varpi_2 + \frac{\mathrm{d}V}{\mathrm{d}t}\,, \quad \varpi_3 + \frac{\mathrm{d}W}{\mathrm{d}t}\,,$$

will be the components of the effective force of the same particle in the second case. Now since by hypothesis $\varpi_1\,\mathrm{d}x + \varpi_2\,\mathrm{d}y + \varpi_3\,\mathrm{d}z$ is an exact differential, as follows from Equation C, and U, V, W, are functions of t only, it follows at once that

$$\left(\varpi_1 + \frac{\mathrm{d}U}{\mathrm{d}t}\right)\mathrm{d}x + \left(\varpi_2 + \frac{\mathrm{d}V}{\mathrm{d}t}\right)\mathrm{d}y + \left(\varpi_3 + \frac{\mathrm{d}W}{\mathrm{d}t}\right)\mathrm{d}z$$

is an exact differential, where x, y, z, are the co-ordinates of any particle referred to the old axes, which are themselves moving in space with velocities U, V, W. But if x_1, y_1, z_1, be the co-ordinates of the same particle referred to parallel axes fixed in space, we have

$$x_1 = x + \int U\,\mathrm{d}t\,, \quad y_1 = y + \int V\,\mathrm{d}t\,, \quad z_1 = z + \int W\,\mathrm{d}t\,,$$

whence, supposing the time constant, $\mathrm{d}x = \mathrm{d}x_1$, $\mathrm{d}y = \mathrm{d}y_1$, $\mathrm{d}z = \mathrm{d}z_1$, and therefore

$$\left(\varpi_1 + \frac{\mathrm{d}U}{\mathrm{d}t}\right)\mathrm{d}x_1 + \left(\varpi_2 + \frac{\mathrm{d}V}{\mathrm{d}t}\right)\mathrm{d}y_1 + \left(\varpi_3 + \frac{\mathrm{d}W}{\mathrm{d}t}\right)\mathrm{d}z_1$$

is an exact differential. Hence, Equation A can be satisfied by a suitable value of p. Denoting by p the pressure about the particle whose co-ordinates are x, y, x, in the first case, the pressure about the same particle in the second case will be

$$p + \psi(t) - \rho \cdot \left\{\frac{\mathrm{d}U}{\mathrm{d}t}x + \frac{\mathrm{d}V}{\mathrm{d}t}y + \frac{\mathrm{d}W}{\mathrm{d}t}z\right\}\,,$$

none of the terms of which will alter abruptly, since by hypothesis p does not. Since then the present hypothesis satisfies all the requisite conditions, it follows from Art. 2, page 18 that this hypothesis is correct. If $\mathscr{F}$ be the additional effective force of any particle of the vessel in consequence of the motion of translation, and we take new axes of x', y', z', of which the first is in the direction of $\mathscr{F}$, the additional term introduced into the value of the pressure will be $-\rho\,\mathscr{F}\,x'$, omitting the arbitrary function of the time. The resultant of the additional pressures on the sides of the vessel will be equal to $\mathscr{F}$ multiplied by the mass of the fluid, and will pass through the centre of gravity of the fluid, and act in the direction of $-x'$.

4.1. Motion between two cylindrical surfaces having a common axis

8. Let us conceive a mass of fluid at rest, bounded by two cylindrical surfaces having a common axis, these surfaces being either infinite or bounded by two planes perpendicular to their axis. Let us suppose the several generating lines of these cylindrical surfaces to be moved parallel to themselves in any given manner consistent with the condition that the volume of the fluid be not altered: it is required to determine the initial motion at any point of the mass.
Since the motion will take place in two dimensions, let the fluid be referred to polar co-ordinates r, θ, in a plane perpendicular to the axis, r being measured from the axis. Let a be the radius of the inner surface, b that of the outer, $f(\theta)$ the normal velocity of any point of the inner surface, $F(\theta)$ the corresponding quantity for the outer.
Since for any particular radius vector between a and b the value of ϕ is a periodic function of θ which does not become infinite, (for the motion at each point of each bounding surface is supposed to be finite), and which does not alter abruptly, it may be expanded in a converging series of sines and cosines of θ and its multiples. Let then

$$\phi = P_0 + \sum_{1}^{\infty} \left[P_n \cdot \cos(n\theta) + Q_n \cdot \sin(n\theta) \right] . \tag{4.1}$$

Substituting the above value in the equation

$$r \cdot \frac{\mathrm{d}}{\mathrm{d}r} \left(r \cdot \frac{\mathrm{d}\phi}{\mathrm{d}r} \right) + \frac{\mathrm{d}^2\phi}{\mathrm{d}\theta^2} = 0 \,, \tag{4.2}$$

which ϕ is to satisfy, and equating to zero the coefficients of corresponding sines and cosines, which is allowable, since a given function can be expanded in only one series of the form according to Equation 4.1, we find that P_0 must satisfy the equation

$$r \cdot \frac{\mathrm{d}}{\mathrm{d}r} \left(r \cdot \frac{\mathrm{d}P_0}{\mathrm{d}r} \right) = 0 \,,$$

of which the general integral is

$$P_0 = A \cdot \ln(r) + B \,,$$

the base being e,[1] and P_n and Q_n must both satisfy the same equation, viz.

$$r \cdot \frac{\mathrm{d}}{\mathrm{d}r} \cdot \left(r \cdot \frac{\mathrm{d}P_n}{\mathrm{d}r} \right) - n^2 P_n = 0 \,,$$

of which the general integral is

$$P_n = \frac{C}{r^n} + C' r^n \,.$$

We have then, omitting the arbitrary constant in ϕ, as will be done for the future, since we have occasion to use only the differential coefficients of ϕ,

$$\phi = A_0 \cdot \ln(r) + \sum_1^\infty \left\{ \left(\frac{A_n}{r^n} + A'_n r^n \right) \cdot \cos(n\theta) + \left(\frac{B_n}{r^n} + B'_n r^n \right) \cdot \sin(n\theta) \right\} \,, \quad (4.3)$$

with the conditions

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = f(\theta) \text{ when } \; r = a \,, \quad (4.4)$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = F(\theta) \text{ when } \; r = b \,. \quad (4.5)$$

Let

$$f(\theta) = C_0 + \sum_1^\infty \left[C_n \cdot \cos(n\theta) + D_n \cdot \sin(n\theta) \right] \,,$$

$$F(\theta) = C'_0 + \sum_1^\infty \left[C'_n \cdot \cos(n\theta) + D'_n \cdot \sin(n\theta) \right] \,;$$

so that

$$C_0 = \frac{1}{2\pi} \cdot \int_0^{2\pi} f(\theta') \, \mathrm{d}\theta' \,,$$

$$C_n = \frac{1}{\pi} \cdot \int_0^{2\pi} f(\theta') \cdot \cos(n\theta') \, \mathrm{d}\theta' \,,$$

$$D_n = \frac{1}{\pi} \cdot \int_0^{2\pi} f(\theta') \cdot \sin(n\theta') \, \mathrm{d}\theta' \,,$$

[1] In the original paper STOKES uses the notation $\log(x)$. In the following we always use the notation $\ln(x)$, the *Logarithmus naturalis*. It is further

$$\log_e(x) = \frac{\log_{10}(x)}{\log_{10}(e)} = \ln(x) \,.$$

with similar expressions for C'_0, &c. Then the condition according to Equation 4.4 gives

$$\begin{aligned}
&\frac{A_0}{a} + \sum_1^\infty n \cdot \left\{ \left(-\frac{A_n}{a^{n+1}} + A'_n a^{n-1} \right) \cdot \cos(n\theta) \right. \\
&\left. + \left(-\frac{B_n}{a^{n+1}} + B'_n a^{n-1} \right) \cdot \sin(n\theta) \right\} \\
&= C_0 + \sum_1^\infty \left(C_n \cdot \cos(n\theta) + D_n \cdot \sin(n\theta) \right) ;
\end{aligned}$$

whence,

$$\begin{aligned}
A_0 &= a \cdot C_0 \,, \\
\frac{A_n}{a^{n+1}} - A'_n a^{n-1} &= -\frac{1}{n} \cdot C_n \,, \\
\frac{B_n}{a^{n+1}} - B'_n a^{n-1} &= -\frac{1}{n} \cdot D_n \,.
\end{aligned}$$

Similarly, from the condition according to Equation 4.5, we get

$$\begin{aligned}
A_0 &= b \cdot C'_0 \,, \\
\frac{A_n}{b^{n+1}} - A'_n b^{n-1} &= -\frac{1}{n} \cdot C'_n \,, \\
\frac{B_n}{b^{n+1}} - B'_n b^{n-1} &= -\frac{1}{n} \cdot D'_n \,.
\end{aligned}$$

It will be observed that $a \cdot C_0 = b \cdot C'_0$, by the condition that the volume of fluid remains unchanged, which gives

$$a \cdot \int_0^{2\pi} f(\theta') \, \mathrm{d}\theta' = b \cdot \int_0^{2\pi} F(\theta') \, \mathrm{d}\theta' \,.$$

From the above equations we easily get

$$A_n = \frac{a^{2n} b^{2n}}{n \cdot (b^{2n} - a^{2n})} \cdot \left\{ b^{-n+1} \cdot C'_n - a^{-n+1} \cdot C_n \right\} \,,$$

and, changing the sign of n,

$$A'_n = \frac{1}{n \cdot (b^{2n} - a^{2n})} \cdot \left\{ b^{n+1} \cdot C'_n - a^{n+1} \cdot C_n \right\} \,,$$

with similar expressions for B_n and B'_n, involving D in place of C.
We have then

$$\begin{aligned}\phi = a \cdot C_0 \cdot \ln(r) + \sum_1^\infty \frac{1}{n} \cdot \frac{1}{(b^{2n} - a^{2n})} \\ \cdot \Big\{ \big[\big(b^{-n+1} \cdot C'_n - a^{-n+1} \cdot C_n\big) \cdot \cos(n\theta) \\ + \big(b^{-n+1} \cdot D'_n - a^{-n+1} \cdot D_n\big) \cdot \sin(n\theta) \big] \cdot \frac{a^{2n} b^{2n}}{r^n} \\ + \big[\big(b^{n+1} \cdot C'_n - a^{n+1} \cdot C_n\big) \cdot \cos(n\theta) \\ + \big(b^{n+1} \cdot D'_n - a^{n+1} \cdot D_n\big) \cdot \sin(n\theta) \big] \cdot r^n \Big\} \ , \end{aligned} \tag{4.6}$$

which completely determines the motion.
It will be necessary however, (Art. 2., page 18), to shew that this value of ϕ does not alter abruptly for points within the fluid, as may be easily done. For the quantities C_n, D_n cannot be greater than $\frac{1}{\pi} \cdot \int_0^{2\pi} \pm f(\theta)\, \mathrm{d}\theta$, where each element of the integral is taken positively; and since by hypothesis $f(\theta)$ is finite for all values of θ from 0 to 2π, it follows that neither C_n nor D_n can be numerically greater than a constant quantity which is independent of n. The same will be true of C'_n and D'_n. Remembering then that $r > a$ and $r < b$, it can be easily shewn that the series which occur in Equation 4.6 have their terms numerically less than those of eight geometric series respectively whose ratios are less than unity; and since moreover the terms of the former set of series do not alter abruptly, it follows that ϕ cannot alter abruptly. The same may be proved in a similar manner of the differential coefficients of ϕ. The other infinite series expressing the value of ϕ which occur in this paper may be treated in the same way: and in Art. 10., page 41, where ϕ is expressed by a definite integral, the value of ϕ and its differential coefficients will alter continuously, since that is the case with each element of the integral. It will be unnecessary therefore to refer again to the condition according to Equation d.

If the fluid be infinitely extended, we must suppose C'_n and D'_n to vanish in Equation 4.6, since the velocity vanishes at an infinite distance; we must then make b infinite, which reduces the above equation to

$$\phi = aC_0 \cdot \ln(r) - \sum_1^\infty \frac{a^{n+1}}{n \cdot r^n} \cdot \{C_n \cdot \cos(n\theta) + D_n \cdot \sin(n\theta)\} \ . \tag{4.7}$$

This value of ϕ may be put under the form of a definite integral: for, replacing C_0, C_n and D_n by their values, it becomes

$$\frac{a}{2\pi} \cdot \ln(r) \cdot \int_0^{2\pi} f(\theta')\, \mathrm{d}\theta' - \frac{a}{\pi} \cdot \sum_1^\infty \frac{1}{n} \left(\frac{a}{r}\right)^n \cdot \int_0^{2\pi} f(\theta') \cdot \cos\left[n \cdot (\theta - \theta')\right] \mathrm{d}\theta' \, ,$$

which becomes on summing the series

$$\frac{a}{2\pi} \cdot \ln(r) \cdot \int_0^{2\pi} f(\theta')\,\mathrm{d}\theta' + \frac{a}{\pi} \cdot \int_0^{2\pi} \ln \sqrt{1 - 2 \cdot \frac{a}{r} \cdot \cos(\theta - \theta') + \frac{a^2}{r^2}} \cdot f(\theta')\,\mathrm{d}\theta' ;$$

whence

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{a}{\pi r} \cdot \int_0^{2\pi} \left\{ \frac{1}{2} + \frac{ar \cdot \cos(\theta - \theta') - a^2}{r^2 - 2ar \cdot \cos(\theta - \theta') + a^2} \right\} \cdot f(\theta')\,\mathrm{d}\theta' .$$

If we suppose r to become equal to a, the quantity under the integral sign vanishes, except for values of θ', which are indefinitely near to θ. The value of the integral itself becomes $\pi \cdot f(\theta)$.[2] Hence it appears, that to the disturbance of each element of the surface, there corresponds a normal velocity of the particles in contact with the surface, which is zero, except just about the disturbed element. The whole disturbance of the fluid will be the aggregate of the disturbances due to those of the several elements of the surface. The case of the initial motion of fluid within a cylinder, and the analogous cases of motion within and without a sphere, which will be given in the next article, may be treated in the same manner.
The velocity in the direction of r given by Equation 4.7, $(= \frac{\mathrm{d}\phi}{\mathrm{d}r})$,

$$= \frac{a \cdot C_0}{r} + \sum_1^\infty \left(\frac{a}{r}\right)^{n+1} \cdot \{C_n \cdot \cos(n\theta) + D_n \cdot \sin(n\theta)\} ,$$

and that perpendicular to r, and reckoned positive in the same direction as θ, $(= \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}\theta})$,

$$\sum_1^\infty \left(\frac{a}{r}\right)^{n+1} \cdot \{C_n \cdot \sin(n\theta) - D_n \cdot \cos(n\theta)\} .$$

Conceive a mass of fluid comprised between two infinite parallel planes, and suppose that a certain portion of this fluid contains solid bodies bounded by cylindrical surfaces perpendicular to these planes. The whole being at first at rest, suppose that the surfaces of these solids are moved in any manner, the motion being in two dimensions. Conceive a circular cylindrical surface described perpendicular to the parallel planes, and with a radius so large that all the solids are comprised within it. Then, (Art. 4., page 22), we may suppose the motion of the fluid at any time to have been produced directly by impact. On this supposition the initial motion of the part of the fluid without the above cylindrical surface will be determined in terms of the normal motion of the fluid forming that surface, as has just been done. If C_0 be different from zero, then, at a great distance in the fluid, the velocity will be ultimately $\frac{a \cdot C_0}{r}$, and directed to or from the axis of the cylinder, and alike in

[2] Poisson, *Théorie de mathématique la Chaleur*, chap. VII, Paris 1835, `https://books.google.de/books?id=uvc_AAAAYAAJ&hl=de&source=gbs_navlinks_s`

all directions. Since the rate of increase of volume of a length l of the cylinder is equal to

$$la \cdot \int_0^{2\pi} f(\theta')\,\mathrm{d}\theta' = 2\pi laC_0\,,$$

it appears that the velocity at a great distance is proportional to the expansion or contraction of a unit of length of the solids. If however there should be no expansion or contraction, or if the expansion of some of the solids should make up for the contraction of the rest, then in general the most important part of the motion at a great distance will consist of a velocity $\frac{C'\cos(\theta_1)}{r^2}$ directed to or from the centre, and another $\frac{C'\sin(\theta_1)}{r^2}$ perpendicular to the radius vector, the value of C' and the direction from which θ_1 is measured varying from one instant to another. The resultant of these velocities will vary inversely as the square of the distance.

Resuming the value of ϕ given by Equation 4.6, let us suppose that the interior cylindrical surface is rigid, and moved with a velocity C in the direction from which θ is measured, the outer surface being at rest: then $f(\theta) = C \cdot \cos(\theta)$, $F(\theta) = 0$; whence $C_1 = C$, and the other coefficients are each zero. We have then

$$\phi = -\frac{C \cdot a^2}{(b^2 - a^2)} \cdot \left\{ \frac{b^2}{r} + r \right\} \cdot \cos(\theta)\,. \tag{4.8}$$

Suppose now that the inner cylinder has a small oscillatory motion about an axis parallel to the axes of the cylinders, the cylinders having their axes coincident in the position of equilibrium. Let ψ be the angle which a plane drawn through the axis of rotation, and that of the solid cylinder at any time makes with a vertical plane drawn through the former. The motion of translation of the axis of the cylinder will differ from a rectilinear motion by quantities depending on ψ^2: the motion of rotation about its axis will be of the order ψ, but will have no effect on the fluid. Therefore in considering the motion of the fluid we may, if we neglect squares of ψ, consider the motion of the cylinder rectilinear. The expression given for ϕ by Equation 4.8 will be accurately true only for the instant when the axes of the cylinders coincide; but since the whole resultant pressure on the solid cylinder in consequence of the motion is of the order ψ, we may, if we neglect higher powers of ψ than the first, employ the approximate value of ϕ given by Equation 4.8. Neglecting the square of the velocity, we have

$$p = -\rho \cdot \frac{\mathrm{d}\phi}{\mathrm{d}t}\,.$$

In finding the complete value of $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ it would be necessary to express ϕ by co-ordinates referred to axes fixed in space, which after differentiation we might suppose to coincide with others fixed in the body. But the additional terms so introduced depending on the square of the velocity, which by hypothesis is neglected, we may

differentiate the value of ϕ given by Equation 4.8 as if the axes were fixed in space. We have then, to the first order of approximation,

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{a^2 \cdot \frac{\mathrm{d}C}{\mathrm{d}t}}{(b^2 - a^2)} \cdot \left\{ \frac{b^2}{r} + r \right\} \cdot \cos(\theta)\,.$$

If l be the length of the cylinder, the pressure on the element $la \cdot \mathrm{d}\theta$, resolved parallel to x and reckoned positive when it acts in the direction of x,

$$= -\frac{\rho l a^3 \cdot \frac{\mathrm{d}C}{\mathrm{d}t}}{(b^2 - a^2)} \cdot \left\{ \frac{b^2}{a} + a \right\} \cdot \cos^2(\theta)\,\mathrm{d}\theta\,;$$

and integrating from $\theta = 0$ to $\theta = 2\pi$, we have the whole resultant pressure parallel to x

$$= -\frac{b^2 + a^2}{(b^2 - a^2)} \cdot \pi \rho l a^2 \cdot \frac{\mathrm{d}C}{\mathrm{d}t}\,.$$

Since $\frac{\mathrm{d}C}{\mathrm{d}t}$ is the effective force of the axis, parallel to x, and that parallel to y is of the order ψ^2, we see that the effect of the inertia of the fluid is to increase the mass of the cylinder by $\frac{b^2+a^2}{(b^2-a^2)} \cdot \mu$, where μ is the mass of the fluid displaced. This imaginary additional mass must be supposed to be collected at the axis of the cylinder.

If the cylinder oscillates in an infinitely extended fluid, it follows that $b = \infty$, and the additional mass becomes equal to that of the fluid displaced. This appears to be a result capable of being compared with experiment, though not with very great accuracy. Two cylinders of the same material, and of the same radius, but whose lengths differ by several radii, might be made to oscillate in succession in a fluid, at a depth sufficiently great to allow us to neglect the motion of the surface of the fluid. The time of oscillation of each might then be calculated as if the cylinder oscillated in vacuum, acted on by a moving force equal to its weight *minus* that of the fluid displaced, acting downwards through its centre of gravity, and having its mass increased by an unknown mass collected in the axis. Equating the time of oscillation so calculated to that given by observation, we should determine the unknown mass. The difference of these masses would be very nearly equal to the mass which must be added to that of a cylinder whose length is equal to the difference of the lengths of the first two, when the motion is in two dimensions. This evidently comes to supposing that, at a distance from the middle of the longer cylinder not greater than half the difference of the lengths of the two, the motion may be taken as in two dimensions. The ends of the cylinders may be of any form, provided that they are all of the same. They may be suspended by fine equal wires, in which case we should have a compound pendulum, or attached to a rigid body oscillating above the fluid by means of thin flat bars of metal, whose plane is in the plane of motion. Another way of getting rid of the motion in three dimensions about the ends would be, to make those ends plane, and to fix two rigid planes parallel to the plane of motion, which should be almost in contact with the ends of

the cylinder.

4.2. Motion between two concentric spherical surfaces.–Motion of a ball pendulum enclosed in a spherical case

9. Let a mass of fluid be at rest, comprised between two concentric spherical surfaces. Let the several points of these surfaces be moved in any manner consistent with the condition that the volume of the fluid be not changed: it is required to determine the initial motion at any point of the mass.
Let a, b, be the radii of the inner and outer spherical surfaces respectively; then employing the co-ordinates r, θ, ω, where r is the distance from the centre, θ the angle which r makes with a fixed line passing through the centre, and ω the angle which a plane passing through these two lines makes with a fixed plane through the latter. The value of ϕ corresponding to any radius vector comprised between a and b can be expanded in a converging series of LAPLACE's coefficients. Let then

$$\phi = V_0 + V_1 + \cdots + V_n + \cdots ,$$

V_n being a LAPLACE's coefficient of the n^{th} order.
Substituting in the equation,

$$r \cdot \frac{\mathrm{d}^2 r\phi}{\mathrm{d}r^2} + \frac{1}{\sin(\theta)} \cdot \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin(\theta) \cdot \frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right) + \frac{1}{\sin^2(\theta)} \cdot \frac{\mathrm{d}^2\phi}{\mathrm{d}\omega^2} = 0 ,$$

which ϕ is to satisfy, employing the equation

$$n \cdot (n+1) \cdot V_n + \frac{1}{\sin(\theta)} \cdot \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin(\theta) \cdot \frac{\mathrm{d}V_n}{\mathrm{d}\theta}\right) + \frac{1}{\sin^2(\theta)} \cdot \frac{\mathrm{d}^2 V_n}{\mathrm{d}\omega^2} = 0 , \tag{4.9}$$

and then equating to zero the LAPLACE's coefficients of the several orders, we find

$$r \cdot \frac{\mathrm{d}^2 r V_n}{\mathrm{d}r^2} - n \cdot (n+1) \cdot V_n = 0 .$$

The general integral of this equation is

$$V_n = C r^n + \frac{C'}{r^{n+1}} ,$$

where C and C' are functions of θ and ω. Substituting in Equation 4.9, and equating coefficients of the two powers of r which enter into it separately to zero, we find that both C and C' satisfy it, and therefore are both LAPLACE's coefficients

of the n^{th} order. We have then

$$\phi = \sum_0^\infty \left(Y_n r^n + \frac{Z_n}{r^{n+1}} \right) , \tag{4.10}$$

where Y_n and Z_n are each LAPLACE's coefficients of the n^{th} order, and do not contain r. Let $f(\theta, \omega)$ be the normal velocity of the point of the inner surface corresponding to θ and ω, $F(\theta, \omega)$ the corresponding quantity for the outer; then the conditions which ϕ is to satisfy are that

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = f(\theta, \omega) \text{ when } \ r = a \,,$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = F(\theta, \omega) \text{ when } \ r = b \,.$$

Let $f(\theta, \omega)$, expanded in a series of LAPLACE's coefficients, be

$$P_0 + P_1 + \cdots + P_n + \cdots$$

which expansion may be performed by the usual formula, if not by inspection: then the first condition gives

$$\sum_0^\infty \left(n Y_n a^{n-1} - \frac{(n+1) \cdot Z_n}{a^{n+2}} \right) = \sum_0^\infty P_n \,;$$

and equating LAPLACE's coefficients of the same order, we get

$$n \cdot Y_n a^{n-1} - \frac{(n+1) \cdot Z_n}{a^{n+2}} = P_n \tag{4.11}$$

Let $F(\theta, \omega)$, expanded in a series of LAPLACE's coefficients, be

$$P'_0 + P'_1 + \cdots + P'_n + \cdots \,;$$

then from the second condition, we get

$$n \cdot Y_n b^{n-1} - \frac{(n+1) \cdot Z_n}{b^{n+2}} = P'_n \tag{4.12}$$

From Equation 4.11 and Equation 4.12 we easily get

$$Y_n = \frac{P'_n \cdot b^{n+2} - P_n \cdot a^{n+2}}{n \cdot (b^{2n+1} - a^{2n+1})} \,,$$

$$Z_n = \frac{a^{2n+1}b^{2n+1} \cdot \left\{ \frac{P'_n}{b^{n-1}} - \frac{P_n}{a^{n-1}} \right\}}{(n+1) \cdot (b^{2n+1} - a^{2n+1})},$$

provided n be greater than 0. If $n = 0$, we have

$$-\frac{Z_0}{a^2} = P_0\,, \quad -\frac{Z_0}{b^2} = P'_0\,.$$

But the condition that the volume of the fluid be not altered, gives

$$a^2 \cdot \int_0^{\pi} \int_0^{2\pi} f(\theta,\omega) \cdot \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\omega = b^2 \cdot \int_0^{\pi} \int_0^{2\pi} F(\theta,\omega) \cdot \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\omega\,,$$
$$\text{or} \quad 4\pi a^2 P_0 = 4\pi b^2 P'_0\,,$$

which reduces the two equations just given to one.
We have then, omitting the constant Y_0,

$$\phi = -\frac{P_0 \cdot a^2}{r} + \sum_1^{\infty} \frac{1}{(b^{2n+1} - a^{2n+1})} \cdot \left\{ \frac{1}{n} \cdot \left(P'_n \cdot b^{n+2} - P_n \cdot a^{n+2} \right) \cdot r^n \right.$$
$$\left. + \frac{a^{2n+1} \cdot b^{2n+1}}{(n+1)} \cdot \left(\frac{P'_n}{b^{n-1}} - \frac{P_n}{a^{n-1}} \right) \cdot \frac{1}{r^{n+1}} \right\} \tag{4.13}$$

which determines the motion.
When the fluid is infinitely extended, we have $P'_n = 0$, since the velocity vanishes at an infinite distance, and $b = \infty$, whence

$$\phi = -\frac{P_0 \cdot a^2}{r} - \sum_1^{\infty} \frac{a^{n+2} \cdot P_n}{(n+1) \cdot r^{n+1}}\,.$$

It may be proved, precisely as was done, (Art. 8., page 29), for a motion in two dimensions, that if any portion of an infinitely extended fluid be disturbed by the motion of solid bodies, or otherwise, if all the fluid beyond a certain distance from the part disturbed were at first at rest, the velocity at a great distance will ultimately be directed to or from the disturbed part, and will be the same in all directions, and will vary as $\frac{1}{r^2}$. The coefficient of $\frac{1}{r^2}$ will be proportional to the rate of gain or loss of volume of the part disturbed. If however this rate should be zero, then the most important part of the velocity at a great distance will in general be that depending on the term $-\frac{a^3 \cdot P_1}{2 \cdot r^2}$ in ϕ. Since the general form of P_1 is

$$A \cdot \cos(\theta) + B \cdot \sin(\theta)\cos(\omega) + C \cdot \sin(\theta)\sin(\omega)\,,$$

we easily find, by making use of rectangular co-ordinates, changing the direction of the axes, and then again adopting polar co-ordinates, that the above term in ϕ takes the form $\frac{D \cdot \cos(\theta_1)}{r^2}$, θ_1 being measured from some line passing through the

origin. The motion will therefore be the same as that round a ball pendulum in an incompressible fluid, the centre of the ball being in the origin; a case of motion which will be considered immediately. In order to represent the motion at different times, we must suppose the velocity and direction of motion of the ball to change with the time.

The value of ϕ given by Equation 4.13 is applicable to the determination of the motion of a ball pendulum enclosed in a spherical case which is concentric with the ball in its position of equilibrium. If C be the velocity of the centre of the ball at the instant when the centres of the ball and case coincide, and if θ be measured from the direction in which it is moving, we shall have

$$f(\theta) = C \cdot \cos(\theta)\,, \quad F(\theta) = 0\,;$$
$$\therefore P_0 = 0\,, \quad P_1 = C \cdot \cos(\theta)\,, \quad P_2 = 0\,, \;\&c., \quad P_0' = 0\,, \;\&c.,$$

and the value of ϕ for this instant is accurately

$$-\frac{C \cdot a^3}{(b^3 - a^3)} \cdot \left(r + \frac{b^3}{2r^2}\right) \cdot \cos(\theta)\,,$$

which, when $b = \infty$, becomes

$$-\frac{Ca^3 \cdot \cos(\theta)}{2r^2}\,,$$

which is the known expression for the value of ϕ for a sphere oscillating in an infinitely extended, incompressible fluid.

It may be shewn, by precisely the same reasoning as was employed in the case of the cylinder, that in calculating the small oscillations of the sphere the value of $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ to be employed is

$$-\frac{a^3 \cdot \frac{\mathrm{d}C}{\mathrm{d}t}}{(b^3 - a^3)} \cdot \left(a + \frac{b^3}{2a^2}\right) \cdot \cos(\theta)\,;$$

and from the equation $p = -\rho \cdot \frac{\mathrm{d}\phi}{\mathrm{d}t}$, we easily find that the whole resultant pressure on the sphere in the direction of its centre, and tending to retard it, is

$$\frac{4}{3} \cdot \frac{\pi\rho a^5}{(b^3 - a^3)} \cdot \left(a + \frac{b^3}{2a^2}\right) \frac{\mathrm{d}C}{\mathrm{d}t}\,,$$

and that perpendicular to this direction is zero. Since $\frac{\mathrm{d}C}{\mathrm{d}t}$ is the effective force of the centre in the direction of the motion, and that perpendicular to this direction is of the second order, the effect of the inertia of the fluid will be to increase the mass of the sphere by a mass

$$= \frac{4}{3} \cdot \frac{\pi\rho a^5}{(b^3 - a^3)} \cdot \left(a + \frac{b^3}{2a^2}\right) = \frac{b^3 + 2a^3}{(b^3 - a^3)} \cdot \frac{\mu}{2}\,,$$

μ being the mass of the fluid displaced; so that the effect of the case is, to increase the mass which we must suppose to be added to that of the ball in the ratio of $(b^3 + 2a^3)$ to $(b^3 - a^3)$.
Poisson, in his solution of the problem of the oscillating sphere given in the *Mémoires de l'Académie*[3] arrives at a different conclusion, viz. that the case does not at all affect the motion of the sphere. When the elimination which he proposes at p. 563 is made, the last term of equation (f) p. 550 becomes

$$\frac{\delta\gamma}{2a^2c\lambda \cdot (l - \delta\gamma)} \cdot \left(\frac{\mathrm{d}^3\zeta}{\mathrm{d}t^3} + \frac{\mathrm{d}^3\zeta'}{\mathrm{d}t^3}\right),$$

where a is the velocity of propagation of sound, and δ the ratio of the density of air to that of the ball, ζ and ζ' being functions derived from others which enter into the value of ϕ by putting $r = c$, where c is the radius of the ball. He then argues that this term may be neglected as insensible, since it involves δ in the numerator and a^2 in the denominator, tacitly assuming that $\frac{\mathrm{d}^3\zeta}{\mathrm{d}t^3} + \frac{\mathrm{d}^3\zeta'}{\mathrm{d}t^3}$ is not large since ϕ is not large. Now for the disturbances of the air which have the same period as those of the pendulum, $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ is not large compared with ϕ, as it is for those on which sound depends. Let then Poisson's solution of equation (a), p. 547 of the volume already mentioned, be put under the form

$$\phi = \frac{1}{r^2} \cdot \left\{f\left(t - \frac{r}{a}\right) + F\left(t + \frac{r}{a}\right)\right\} + \frac{1}{ar} \cdot \left\{f'\left(t - \frac{r}{a}\right) - F'\left(t + \frac{r}{a}\right)\right\},$$

f' and F' denoting the derived functions, and all the Laplace's coefficients except those of the first order being omitted, the value of ϕ just given being supposed to be a Laplace's coefficient of that order. Then if we expand the above functions in series ascending according to powers of $\frac{r}{a}$, we find

$$\phi = \frac{1}{r^2} \cdot \{f(t) + F(t)\} - \frac{1}{2a^2} \cdot \{f''(t) + F''(t)\} + \frac{1}{3a^3} \cdot \{f'''(t) - F'''(t)\} + \cdots;$$

and in order that when $a = \infty$ this equation may coincide with Equation 4.10. When all the Laplace's coefficients except those of the first order are omitted in that equation, it will be seen that it is necessary to suppose $f'''(t) - F'''(t)$, and therefore $f(t) - F(t)$, to be of the order a^3, while $f(t) + F(t)$ is not large. Putting then

$$f(t) = \chi(t) + a^3 \cdot \varpi(t),$$
$$F(t) = \chi(t) - a^3 \cdot \varpi(t),$$

[3] "*Mémoire sur les mouvements simultanés d'un pendule et de l'air environnant*" in Mémoires de l'Académie des sciences, Tome 11, p. 521, 1832. `https://gallica.bnf.fr/ark:/12148/bpt6k3226g`

we shall have

$$\zeta + \zeta' = \chi\left(t - \frac{c}{a}\right) + \chi\left(t + \frac{c}{a}\right) + a^3 \cdot \left\{\varpi\left(t - \frac{c}{a}\right) - \varpi\left(t + \frac{c}{a}\right)\right\};$$

so that $\frac{\mathrm{d}^3(\zeta+\zeta')}{\mathrm{d}t^3}$ contain a term of the order a^2, and the term which Poisson proposes to leave out will be of the same order of magnitude as those retained.
In making the experiment of determining the resistance of the air to an oscillating sphere, it would appear to be desirable to enclose the sphere in a concentric spherical case, which would at the same time exclude currents of air, and facilitate in some measure the experiment by increasing the small quantity which is the subject of observation. The radius of the case however ought not to be nearly as small as that of the ball, for if it were, in the first place a small error in the position of the centre of the ball when at rest might not be insensible, and in the second place the oscillations would have to be inconveniently small, in order that the value of ϕ which has been given might be sufficiently approximate. The effect of a small slit in the upper part of the case, sufficient to allow the wire by which the ball is supported to oscillate, would evidently be insensible, for the condensation being insensible in a vertical plane passing through the axis of rotation, since the alteration of pressure in that plane is insensible, the air would not have a tendency alternately to rush in and out at the slit.

4.3. Effect of a distant rigid plane on the motion of a ball pendulum

10. Although this problem may be more easily solved by an artifice, it may be well to give the direct solution of it by the method mentioned in Article 6., page 27. In order to calculate the motion reflected from the plane, it will be necessary to solve the following problem:

To find the initial motion at any point of a mass of fluid infinitely extended, except where it is bounded by an infinite solid but not rigid plane, the initial motion of each point of the solid plane being given.

It is evident that motion directed to or from a centre situated in the plane, the velocity being the same in all directions, and varying inversely as the square of the distance from that centre, would satisfy the condition that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, and would give to the particles in contact with the plane a velocity directed *along* the plane, except just about the centre. Let us see if the required motion can be made up of an infinite number of such motions directed to or from an infinite number of such centres.
Let x, y, z, be the co-ordinates of any particle of fluid, the plane xy coinciding

with the solid plane, and the axis of z being directed into the fluid. Let x', y', be the co-ordinates of any point in the solid plane: then the part of ϕ corresponding to the motion of the element $\mathrm{d}x'\,\mathrm{d}y'$ of the plane will be

$$\frac{\psi(x',y')\,\mathrm{d}x'\,\mathrm{d}y'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\,,$$

and therefore the complete value of ϕ will be given by the equation

$$\phi=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\psi(x',y')\,\mathrm{d}x'\,\mathrm{d}y'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\,. \tag{4.14}$$

The velocity parallel to z at any point $=\frac{\mathrm{d}\phi}{\mathrm{d}z}$

$$=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\psi(x',y')z\,\mathrm{d}x'\,\mathrm{d}y'}{\sqrt{[(x-x')^2+(y-y')^2+z^2]^3}}\,.$$

Now when z vanishes the quantity under the integral signs vanishes, except for values of x' and y' indefinitely near to x and y respectively, the function $\psi(x',y')$ being supposed to vanish when x' or y' is infinite. Let then $x'=x+\xi$, $y'=y+\eta$, then, $\xi_\prime$ and $\eta_\prime$ being as small as we please, the value of the above expression when $z=0$ becomes

$$-\text{the limit of}\quad\int_{-\xi_\prime}^{\xi_\prime}\int_{-\eta_\prime}^{\eta_\prime}\frac{z\psi(x+\xi,y+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{(\xi^2+\eta^2+z^2)^3}}\quad\text{when}\quad z=0\,.$$

Now if $\psi(x',y')$ does not alter abruptly between the limits $x-\xi_\prime$ and $x+\xi_\prime$ of x', and $y-\eta_\prime$ and $y+\eta_\prime$ of y', the above expression may be replaced by

$$-\psi(x,y)\times\ \text{the limit of}\quad\int_{-\xi_\prime}^{\xi_\prime}\int_{-\eta_\prime}^{\eta_\prime}\frac{z\,\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{(\xi^2+\eta^2+z^2)^3}}\,,$$

$$\text{which is}\quad=-2\pi\cdot\psi(x,y)\,.$$

If now $f(x',y')$ be the given normal velocity of any point (x',y') of the solid plane, the expression for ϕ given by Equation 4.14 may be made to give the required normal velocity of the fluid particles in contact with the solid plane by assuming

$$\psi(x',y')=-\frac{1}{2\pi}\cdot f(x',y')\,,$$

whence

$$\phi=-\frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x',y')\,\mathrm{d}x'\,\mathrm{d}y'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\,.$$

This expression will be true for any point at a finite distance from the plane xy even when $f(x', y')$ does alter abruptly; for we may first suppose it to alter continuously, but rapidly, and may then suppose the rapidity of alteration indefinitely increased: this will not cause the value of ϕ just given to become illusory for points situated without the plane xy.
If it be convenient to use polar co-ordinates in the plane xy, putting $x = q \cdot \cos(\omega)$, $y = q \cdot \sin(\omega)$, $x' = q' \cdot \cos(\omega')$, $y' = q' \cdot \sin(\omega')$ and replacing $f(x', y')$ by $f(q', \omega')$, the equation just given becomes

$$\phi = -\frac{1}{2\pi} \cdot \int_0^{\infty} \int_0^{2\pi} \frac{f(q', \omega') q' \, \mathrm{d}q' \, \mathrm{d}\omega'}{\sqrt{q'^2 + q^2 - 2qq' \cdot \cos(\omega - \omega') + z^2}} \, .$$

To apply this to the case of a sphere oscillating in a fluid perpendicularly to a fixed rigid plane, let a be the radius of the sphere, and let its centre be moving towards the plane with a velocity C at the time t. Then, (Art. 4., page 22), we may calculate the motion as if it were produced directly by impact. Let h be the distance of the centre of the sphere from the fixed plane at the time t, and let the line h be taken for the axis of z, and let r, θ, be the polar co-ordinates of any point of the fluid, r being the distance from the centre of the sphere, and θ the angle between the lines r and h. Then if the fluid were infinitely extended around the sphere we should have

$$\phi = -\frac{Ca^3 \cdot \cos(\theta)}{2r^2} \, . \tag{4.15}$$

The velocity of any particle, resolved in a direction towards the plane, $= \frac{\mathrm{d}\phi}{\mathrm{d}r} \cdot \cos(\theta) - \frac{1}{r} \frac{\mathrm{d}\phi}{\mathrm{d}\theta} \cdot \sin(\theta)$

$$= \frac{C \cdot a^3}{r^3} \cdot \left\{ \cos^2(\theta) - \frac{1}{2} \cdot \sin^2(\theta) \right\} \, .$$

For a particle in the plane xy we have

$$r \cdot \cos(\theta) = h \, , \quad r \cdot \sin(\theta) = q' \, ,$$

and the above velocity becomes

$$\frac{Ca^3 \cdot (2h^2 - q'^2)}{2 \cdot \sqrt{(h^2 + q'^2)^5}} \, .$$

We must now, according to the method explained in Article 6., page 27, suppose the several points of the plane xy moved with the above velocity parallel to z. We have then

$$f(q', \omega') = \frac{Ca^3 \cdot (2h^2 - q'^2)}{2 \cdot \sqrt{(h^2 + q'^2)^5}} \, ;$$

whence, for the motion of the sphere reflected from the plane,

$$\phi = -\frac{C \cdot a^3}{4\pi} \cdot \int_0^\infty \int_0^{2\pi} \frac{(2h^2 - q'^2)\, q'\, \mathrm{d}q'\, \mathrm{d}\omega'}{\sqrt{(h^2 + q'^2)^5} \cdot \sqrt{q^2 + q'^2 - 2qq' \cdot \cos(\omega - \omega') + z^2}} \tag{4.16}$$

We must next find the velocity, corresponding to this value of ϕ, with which the fluid penetrates the surface of the sphere. We have in general

$$z = h - r \cdot \cos(\theta)\,, \quad q = r \cdot \sin(\theta)\,,$$

whence

$$\frac{1}{\sqrt{q^2 + q'^2 - 2qq' \cdot \cos(\omega - \omega') + z^2}} = \frac{1}{\sqrt{h^2 + r^2 + q'^2 - 2hr \cdot \cos(\theta) - 2q'r \cdot \sin(\theta)\cos(\omega - \omega')}}\,.$$

Now supposing the ratio of a to h to be very small, and retaining the most important term, the value of $\frac{\mathrm{d}\phi}{\mathrm{d}r}$ when $r = a$ will be equal to the coefficient of r when ϕ is expanded in a series ascending according to powers of r,

$$= -\frac{C \cdot a^3}{4\pi} \cdot \int_0^\infty \int_0^{2\pi} \frac{(2h^2 - q'^2) \cdot \{h \cdot \cos(\theta) + q' \cdot \sin(\theta)\cos(\omega - \omega')\}\, q'\, \mathrm{d}q'\, \mathrm{d}\omega'}{(h^2 + q'^2)^4}$$

$$= -\frac{1}{2} C a^3 h \cdot \cos(\theta) \cdot \int_0^\infty \frac{(2h^2 - q'^2)\, q'\, \mathrm{d}q'}{(h^2 + q'^2)^4} = -\frac{Ca^3 \cdot \cos(\theta)}{8h^3}\,. \tag{4.17}$$

In order now to determine the motion reflected from the plane and again from the sphere, we must suppose the several points of the sphere to be moved with a normal velocity $\frac{Ca^3 \cdot \cos(\theta)}{8h^3}$, or, which is the same, we must suppose the whole sphere to be moved towards the plane with a velocity $\frac{Ca^3}{8h^3}$. Hence the value of ϕ corresponding to this motion will be given by the equation

$$\phi = -\frac{Ca^6 \cdot \cos(\theta)}{16h^3 r^2}\,. \tag{4.18}$$

For points at a great distance from the centre of the sphere, the motion which is twice reflected will be very small compared with that which is but once reflected. For points close to the sphere however, with which alone we are concerned, those motions will be of the same order of magnitude, and if we take account of the one we must take account of the other.
Putting $q = r \cdot \sin(\theta)$, $z = h - r \cdot \cos(\theta)$ in Equation 4.16, expanding, and retaining

the two most important terms, we have

$$\phi = C \cdot \left(K - \frac{a^3 r \cdot \cos(\theta)}{8h^3} \right) . \tag{4.19}$$

K being a constant, the value of which is not required, and the second term being evidently found by multiplying the quantity at the second side of Equation 4.17 by r. Adding together the parts of ϕ given by Equation 4.15, Equation 4.18 and Equation 4.19, putting $r = a$, replacing C by $\frac{\mathrm{d}C}{\mathrm{d}t}$, and taking for h the value which it has in equilibrium, just as in the case of the oscillating cylinder in Article 8., page 29, we have for the small motion of the sphere

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = K \cdot \frac{\mathrm{d}C}{\mathrm{d}t} - \frac{a}{2} \cdot \left(1 + \frac{3a^3}{8h^3} \right) \cdot \frac{\mathrm{d}C}{\mathrm{d}t} \cdot \cos(\theta) .$$

The resultant of the part of the pressure due to the first term is zero: that due to the second term is greater than if the plane were removed in the ratio of $1 + \frac{3a^3}{8h^3}$ to 1. Consequently, if we neglect quantities of the order $\frac{a^4}{h^4}$, the effect of the inertia of the fluid is, to add a mass equal to $\left(1 + \frac{3}{8} \cdot \frac{a^3}{h^3}\right) \cdot \frac{\mu}{2}$ to that of the sphere, without increasing the moment of inertia of the latter about its diameter. The effect therefore of a large spherical case is eight times as great as that of a tangent plane to the case, perpendicular to the direction of the motion of the ball.

The effect of a distant rigid plane parallel to the direction of motion of an oscillating sphere might be calculated in the same manner, but as the method is sufficiently explained by the first case, it will be well to employ the artifice before alluded to, an artifice which is frequently employed in this subject. It consists in supposing an exactly symmetrical motion to take place on the opposite side of the rigid plane, by which means we may evidently conceive the plane removed.

Let the sphere be oscillating in the direction of the axis of x, the oscillations in this case, as in the last, being so small that they may be taken as rectilinear in calculating the motion of the fluid; and instead of a rigid plane conceive an equal sphere to exist at an equal distance on the opposite side of the plane xy, moving in the same direction and with the same velocity as the actual sphere. Let r, θ, ω, be the polar co-ordinates of any particle measured from the centre of the sphere, θ being the angle between r and a line drawn through the centre parallel to the axis of x, and ω the angle which the plane passing through these lines makes with the plane xz. Let r', θ', ω', be the corresponding quantities symmetrically measured from the centre of the imaginary sphere.

If the fluid were infinite we should have for the motion corresponding to that of the given sphere

$$\phi = -\frac{Ca^3 \cdot \cos(\theta)}{2r^2} . \tag{4.20}$$

The motion reflected from the plane is evidently the same as that corresponding to

the motion of the imaginary sphere in an infinite mass of fluid, for which we have

$$\phi = -\frac{Ca^3 \cdot \cos(\theta')}{2r'^2}\,. \tag{4.21}$$

Now

$$\begin{aligned} &r' \cdot \cos(\theta') = r \cdot \cos(\theta)\,, \\ &r' \cdot \sin(\theta')\sin(\omega') = r \cdot \sin(\theta)\sin(\omega)\,, \\ &r' \cdot \sin(\theta')\cos(\omega') + r \cdot \sin(\theta)\cos(\omega) = 2h\,; \end{aligned}$$

whence

$$r'^2 = r^2 + 4h^2 - 4hr \cdot \sin(\theta)\cos(\omega)\,,$$

and Equation 4.21 is reduced to

$$\phi = -\frac{Ca^3 r \cdot \cos(\theta)}{2 \cdot \sqrt{\left[r^2 + 4h^2 - 4hr \cdot \sin(\theta)\cos(\omega)\right]^3}}\,.$$

Retaining only the terms of the order $\frac{a^3 r}{h^3}$ or $\frac{r^4}{h^3}$, so as to get the value of $\frac{\mathrm{d}\phi}{\mathrm{d}r}$ to the order $\frac{a^3}{h^3}$, the above equation is reduced to

$$\phi = -\frac{Ca^3 r \cdot \cos(\theta)}{16h^3}\,, \tag{4.22}$$

and the value of $\frac{\mathrm{d}\phi}{\mathrm{d}r}$ when $r = a$ is, to the required degree of approximation,

$$-\frac{Ca^3 \cdot \cos(\theta)}{16h^3}\,.$$

For the value of ϕ corresponding to the motion of the imaginary sphere reflected from the real sphere, we shall therefore have

$$\phi = -\frac{Ca^6 \cdot \cos(\theta)}{32h^3 r^2}\,. \tag{4.23}$$

Adding together the values of ϕ given by Equation 4.20, Equation 4.22 and Equation 4.23, putting $r = a$, and replacing C by $\frac{\mathrm{d}C}{\mathrm{d}t}$, we have, to the requisite degree of approximation,

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{a}{2} \cdot \left(1 + \frac{3}{16} \cdot \frac{a^3}{h^3}\right) \cdot \frac{\mathrm{d}C}{\mathrm{d}t} \cdot \cos(\theta)\,.$$

Hence in this case the motion of the sphere will be the same as if an additional mass equal to $\left(1 + \frac{3}{16} \cdot \frac{a^3}{h^3}\right) \cdot \frac{\mu}{2}$ were collected at its centre. The effect therefore

of a distant rigid plane which is parallel to the direction of the motion of a ball pendulum will be half that of a plane at the same distance, and perpendicular to that direction. It would seem from POISSON's words at page 562 of the eleventh volume of the *Mémoires de l'Académie*, that he supposed the effect in the former case to depend on a higher order of small quantities than that in the latter.
If the ball oscillates in a direction inclined to the plane, the motion may be easily deduced from that in the two cases just given, by means of the principle of superposition.

11. The values of ϕ which have been given for the motion of translation of a sphere and cylinder, do not require us to suppose that either the velocity, or the distance to which the centre of the sphere or axis of the cylinder has been moved, is small, provided the same particles remain in contact with the surface. The same indeed is true of the values corresponding to a motion of translation combined with a motion of contraction or expansion which is the same in all directions, but varies in any manner with the time. The value of ϕ corresponding to a motion of translation of the cylinder is $-\frac{Ca^2\cdot\cos(\theta)}{r}$, C being the velocity of the axis, and θ being measured from a line drawn in the direction of its motion. The whole resultant of the part of the pressure due to the square of the velocity is zero, since the velocity at the point whose co-ordinates are r, θ, is the same as that at the point whose co-ordinates are r and $\pi-\theta$. To find the resultant of the part depending on $\frac{\mathrm{d}\phi}{\mathrm{d}t}$, it will be necessary to express ϕ by means of co-ordinates referred to axes fixed in space. Let Ox, Oy, be rectangular axes passing through the centre of any section of the cylinder, ϖ the angle which the direction of motion of the axis makes with Ox, θ' the inclination of any radius vector to Ox; then

$$\begin{aligned}\phi &= -\frac{Ca^2}{r^2}\cdot\{r\cdot\cos(\theta')\cos(\varpi)+r\cdot\sin(\theta')\sin(\varpi)\}\\ &= -\frac{a^2\cdot(C'x+C''y)}{(x^2+y^2)}\,,\end{aligned}$$

putting C' and C'' for the resolved parts of the velocity C along the axes of x and y respectively. Taking now axes Ax', Ay', parallel to the former and fixed in space, putting α and β for the co-ordinates of O, differentiating ϕ with respect to t, and replacing $\frac{\mathrm{d}\alpha}{\mathrm{d}t}$ by C', and $\frac{\mathrm{d}\beta}{\mathrm{d}t}$ by C'', and then supposing α and β to vanish, we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}=\frac{a^2C^2}{(x^2+y^2)}-\frac{2a^2\cdot(C'x+C''y)^2}{(x^2+y^2)^2}-\frac{a^2\cdot\left(x\cdot\frac{\mathrm{d}C'}{\mathrm{d}t}+y\cdot\frac{\mathrm{d}C''}{\mathrm{d}t}\right)}{(x^2+y^2)}\,.$$

The resultant of the part of the pressure due to the first two terms is zero, since the pressure at the point (x,y) depending on these terms is the same as that at the point $(-x,-y)$. It will be easily found that the resultant of the whole pressure parallel to x, and acting in the negative direction, on a length l of the cylinder,

is equal to $\pi\rho la^2 \cdot \frac{\mathrm{d}C'}{\mathrm{d}t}$, and that parallel to y equal to $\pi\rho la^2 \cdot \frac{\mathrm{d}C''}{\mathrm{d}t}$. The resultant of these two will be $\pi\rho la^2\,\mathscr{F}$, where $\mathscr{F}$ is the effective force of a point in the axis of the cylinder, and will act in a direction opposite to that of $\mathscr{F}$. Hence the only effect of the motion of the fluid will be, to increase the mass of the cylinder by that of the fluid displaced. In a similar manner it may be proved that, when a solid sphere moves in any manner in an infinite fluid, the only effect of the motion of the fluid is to increase the mass of the sphere by half that of the fluid displaced. A similar result may be proved to be true for any solid symmetrical with respect to two planes at right angles to each other, and moving in the direction of the line of their intersection in an infinitely extended fluid, the solid and fluid having been at first at rest. Let the planes of symmetry be taken for the planes of xy and xz, the origin being fixed in the body: then it is evident that the resultant of the pressure on the solid due to the motion will be in the direction of the axis of x, and that there will be no resultant couple. Let C be the velocity of the solid at any time; then the value of ϕ at that time will be of the form $C\psi(x, y, z)$, where C alone contains t, (Art. 4., page 22), and the velocity of the particle whose co-ordinates are x, y, z, being proportional to C, the *vis viva* of the solid and fluid together will be proportional to C^2. Now if no forces act on the fluid and solid, except the pressure of the fluid, this *vis viva* must be constant;[4] therefore C must be constant; therefore the resultant of the fluid pressure on the solid must be zero. If now C be a function of t we shall have

$$p = -\rho \cdot \psi(x, y, z) \cdot \frac{\mathrm{d}C}{\mathrm{d}t} + p' \,,$$

p' being the pressure when C is constant. Since therefore the resultant of the fluid pressure varies for the same solid and fluid as $\frac{\mathrm{d}C}{\mathrm{d}t}$ the effective force, and for

[4]If an incompressible fluid which is homogeneous or heterogeneous, and contains in it any number of rigid bodies, be in motion, the rigid bodies being also in motion, if the rigid bodies are perfectly smooth, and no contacts are formed or broken among them, and if no forces act except the pressure of the fluid, the principle of *vis viva* gives

$$\frac{\mathrm{d}\sum mv^2}{\mathrm{d}t} = 2 \cdot \iint p_{\prime}\nu \,\mathrm{d}S \,, \qquad (\alpha)$$

where v is the whole velocity of the mass m, and the sign $\sum$ extends over the whole fluid and the rigid bodies spoken of, and where $\mathrm{d}S$ is an element of the surface which bounds the whole, $p_{\prime}$ the pressure about the element $\mathrm{d}S$, and ν the normal velocity of the particles in that element, reckoned positive when tending into the fluid, and where the sign $\iint$ extends to all points of the bounding surface. To apply equation Equation α to the case of motion at present considered, let us first confine ourselves to a spherical portion of the fluid, whose radius is r, and whose centre is near the solid, so that $\mathrm{d}S$ refers to the surface of this portion. Let us now suppose r to become infinite: then the second side of Equation α will vanish, provided $p_{\prime}$ remain finite, and ν decreases in a higher ratio than $\frac{1}{r^2}$. Both of these will be true, (Art. 9., page 36); for ν will vary ultimately as $\frac{1}{r^3}$, since there is no alteration of volume. Hence if the sign $\sum$ extend to infinity, we shall have $\sum mv^2$ constant.

different fluids varies as ρ, the effect of the inertia of the fluid will be, to increase the mass of the solid by n times that of the fluid displaced, n depending only on the particular solid considered.

Let us consider two such solids, similar to each other, and having the co-ordinate planes similarly situated, and moving with the same velocities. Let the linear dimensions of the second be greater than those of the first in the ratio of m to 1. Let u, v, w, be the velocities, parallel to the axes, of the particle (x, y, z) in the fluid about the first; then shall the corresponding velocities at the point (mx, my, mz) in the fluid about the second be also u, v, w. For

$$u\,\mathrm{d}(mx) + v\,\mathrm{d}(my) + w\,\mathrm{d}(mz) = m \cdot (u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z) \tag{4.24}$$

and is therefore an exact differential, since $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is one: also the normal at the point (x, y, z) in the first surface will be inclined to the axes at the same angles as the normal at the point (mx, my, mz) of the second surface is inclined to its axes, and therefore the normal velocities of the two surfaces at these points are the same; and the velocities of the fluid at these two points parallel to the axes being also the same, it follows that the normal velocity of each point of the second surface is equal to that of the fluid in contact with it. Lastly, the motion about the first solid being supposed to vanish at an infinite distance from it, that about the second will vanish also. Hence the supposition made with respect to the motion of the fluid about the second surface is correct. Now putting ϕ for $\int (u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z)$ for the fluid in the first case, the corresponding integral for the fluid in the second case will be $m\phi$, if the constant be properly chosen, as follows from Equation 4.24. Consequently the value of that part of the expression for the pressure, on which the resistance depends, will be m times as great for any point in the second case as it is for the corresponding point in the first. Also, each element of the surface of the second solid will be m^2 times as great as the corresponding element of the surface of the first. Hence the whole resistance on the second solid will be m^3 times as great as that on the first, and therefore the quantity n depends only on the *form*, and not on the *size* of the solid.

When forces act on the fluid, it will only be necessary to add the corresponding pressure. Hence when a sphere descends from rest in a fluid by the action of gravity, the motion will be the same as if a moving force equal to that of the sphere *minus* that of the fluid displaced acted on a mass equal to that of the sphere *plus* half that of the fluid displaced. For a cylinder which is so long that we may suppose the length infinite, descending horizontally, everything will be the same, except that the mass to be moved will be equal to that of the cylinder *plus* the whole of the fluid displaced. In these cases, as well as in that of any solid which is symmetrical with respect to two vertical planes at right angles to each other, the motion will be uniformly accelerated, and similar solids of the same material will descend with equal velocities. These results are utterly opposed even to the commonest observation, which shews that large solids descend much more rapidly than small

ones of the same shape and material, and that the velocity of a body falling in a fluid, (such as water), does not sensibly increase after a little time. It becomes then of importance in the theory of resistances to enquire what may be the cause of this discrepancy between theory and observation. The following are the only ways of accounting for it which suggest themselves to me.

First. It has been supposed that the same particles remain in contact with the solid throughout the motion. It must be remembered that we suppose the ultimate molecules of fluids, (if such exist), to be so close that their distance is quite insensible, a supposition of the truth of which there can be hardly any doubt. Consequently we reason on a fluid as if it were infinitely divisible. Now if the motion which takes place in the cases of the sphere and cylinder be examined, supposing for simplicity their motions to be rectilinear, it will be found that a particle in contact with the surface of either moves along that surface with a velocity which at last becomes infinitely small, and that it does not reach the end of the sphere or cylinder from which the whole is moving until after an infinite time, while any particle not in contact with the surface is at last left behind. It seems difficult to conceive of what other kind the motion can be, without supposing a line, (or rather surface) of particles to make an abrupt turn. If it should be said that the particles may come off in tangents, it must be remembered that this sort of motion is included in the condition which has been assumed with respect to the surface.

Secondly. The discrepancy alluded to might be supposed to arise from the friction of the fluid against the surface of the solid. But, for the reason mentioned in the beginning of this paper, this explanation does not appear to me satisfactory.

Thirdly. It appears to me very probable that the *spreading out* motion of the fluid, which is supposed to take place behind the middle of the sphere or cylinder, though dynamically possible, nay, the *only* motion dynamically possible when the conditions which have been supposed are accurately satisfied, is unstable; so that the slightest cause produces a disturbance in the fluid, which accumulates as the solid moves on, till the motion is quite changed. Common observation seems to shew that, when a solid moves rapidly through a fluid at some distance below the surface, it leaves behind it a succession of eddies in the fluid. When the solid has attained its terminal velocity, the product of the resistance, or rather the mean resistance, and any space through which the solid moves, will be equal to half the *vis viva* of the corresponding portion of its *tail of eddies*, so that the resistance will be measured by the *vis viva* in the length of two units of that tail. So far therefore as the resistance which a ship experiences depends on the disturbance of the water, which is independent of its elevation or depression, that ship which leaves the least wake ought, according to this view, to be *cæteris paribus* the best sailer. The resistance on a ship differs from that on a solid in motion immersed in a fluid in the circumstance, that part of the resistance is employed in producing a wave.

Fourthly, the discrepancy alluded to may be due to the mutual friction, or imperfect fluidity of the fluid.

4.4. Motion about an elliptic cylinder of small eccentricity

This particular problem, so far at least as concerns motion of translation, is of little interest in itself, because GREEN (see "*Researches on the Vibrations of Pendulums in Fluid Media*" in Transactions of the Royal Society of Edinburgh, Vol. XIII. p. 54, or p. 315 of his collected works[5]) has determined the motion of a fluid about an ellipsoid moving in any manner with a motion of translation only; and the ellipsoid includes of course as a particular case an elliptic cylinder of any eccentricity. The problem in the text will however serve as an example of the mode of proceeding in the case of a cylinder of any kind differing little from a circular cylinder.
In the case of such a cylinder, supposed to be free from abrupt changes of form, it might safely be assumed that the expression for ϕ which applies to the fluid beyond the greatest radius vector of any point of the surface might also be used for some distance within, as explained in the text. By starting with this assumption, which would be verified in the end, the process of solution would of course be shortened. We should simply have to take the Equation 4.31', form the Equation 4.27 for the velocity normal to the surface, putting $r = c \cdot [1 + \epsilon \cdot \cos(2\theta)]$, and expanding as far as the first power of ϵ, and equate the result to the Equation 4.26. We should thus determine the arbitrary constants in Equation 4.31', which would complete the Solution of the problem.

12. The value of ϕ, which has been deduced, (Art. 8., page 29), for the motion of the fluid about a circular cylinder, is found on the supposition that for each value of r there exists, or may be supposed to exist, a real and finite value of ϕ. This will be true, in any case of motion in two dimensions where $u\,\mathrm{d}x + v\,\mathrm{d}y$ is an exact differential, *for those values of r for which the fluid is not interrupted*, but will be true for values of r for which it is interrupted by solids only when it is possible to replace those solids at any instant by masses of fluid, without affecting the motion of the fluid exterior to them, those masses moving in such a manner that the motion of the whole fluid might have been produced instantaneously by impact. In some cases such a substitution could be made, while in others it probably could not. In any case however we may try whether the expansion given by Equation 4.3 will enable us to get a result, and if it will, we need be in no fear that it is wrong, (Art. 2., page 18). The same remarks will apply to the question of the possibility of the expansion of ϕ in the series of LAPLACE's coefficients given in Equation 4.10, for values of r for which the fluid is interrupted. They will also apply to such a question as that of finding the permanent temperature of the earth due to the solar heat, the earth being supposed to be a homogeneous oblate spheroid, and the points of the surface being supposed to be kept up to constant temperatures, given by observation, depending on the latitude.

[5] `https://www.biodiversitylibrary.org/item/125980`

In cases of fluid motion such as those mentioned, the motion may be determined by conceiving the whole mass of fluid divided into two or more portions, taking the most general value of ϕ for each portion, this value being in general expressed in a different manner for the different portions, then limiting the general value of ϕ for each portion so as to satisfy the conditions with respect to the surfaces of solids belonging to that portion, and lastly introducing the condition that the velocity and direction of motion of each pair of contiguous particles in any two of the portions are the same. The question first proposed will afford an example of this method of solution.
Let an elliptic cylinder be moving with a velocity C, in the direction of the major axis of a section of it made by a plane perpendicular to its axis. The motion being supposed to be in two dimensions, it will be sufficient to consider only this section. Let

$$r = c \cdot [1 + \epsilon \cdot \cos(2\theta)]$$

be the approximate equation to the ellipse so formed, the centre being the pole, and powers of ϵ above the first being neglected. Let a circle be described about the same centre, and having a radius γ equal to $c \cdot (1 + k)$, k being $\not< \epsilon$, and being a small quantity of the order ϵ. Let the portions of fluid within and without the radius γ be considered separately, and putting

$$r = c + z\,.$$

Let the value of ϕ corresponding to the former portion be

$$P + Qz + Rz^2\,,$$

P, Q and R being functions of θ, and the term in z^2 being retained, in order to get the value of $\frac{\mathrm{d}\phi}{\mathrm{d}r}$ true to the order ϵ, while the terms in z^3, &c. are omitted. Substituting this value of ϕ in Equation 4.2, and equating to zero coefficients of different powers of z, we have

$$R = -\frac{Q}{2c} - \frac{1}{2c^2} \cdot \frac{\mathrm{d}^2 P}{\mathrm{d}\theta^2}\,,$$

which is the only condition to be satisfied, since the other equations would only determine the coefficients of z^3, &c. in terms of the preceding ones. We have then

$$\phi = P + Qz - \frac{1}{2c} \cdot \left(Q + \frac{1}{c} \cdot \frac{\mathrm{d}^2 P}{\mathrm{d}\theta^2} \right) \cdot z^2\,. \tag{4.25}$$

Now if ξ be the angle between the normal at any point of the ellipse, and the major axis, we have

$$\xi = \theta + 2\epsilon \cdot \sin(2\theta)\,,$$

and the velocity of the ellipse resolved along the normal

$$= C \cdot \cos(\xi) = C \cdot (1 - \epsilon) \cdot \cos(\theta) + C\epsilon \cdot \cos(3\theta)\,. \tag{4.26}$$

The velocity of the fluid at the same point resolved along the normal is

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} + 2\epsilon \cdot \sin(2\theta) \cdot \frac{1}{r} \cdot \frac{\mathrm{d}\phi}{\mathrm{d}\theta}\,, \tag{4.27}$$

or

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} + \frac{2\epsilon}{c} \cdot \sin(2\theta) \cdot \frac{\mathrm{d}\phi}{\mathrm{d}\theta}\,. \tag{4.27'}$$

Let P and Q be expanded in series of cosines of θ and its multiples, so that

$$P = \sum_0^\infty P_n \cdot \cos(n\theta)\,, \quad Q = \sum_0^\infty Q_n \cdot \cos(n\theta)\,,$$

there being no sines in the expansions of P and Q, since the motion is symmetrical with respect to the major axis; then

$$\phi = \sum_0^\infty \left\{ P_n + Q_n z - \frac{1}{2c} \cdot \left(Q_n - \frac{n^2}{c} \cdot P_n \right) \cdot z^2 \right\} \cdot \cos(n\theta)\,; \tag{4.28}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = \sum_0^\infty \left\{ Q_n - \frac{1}{c} \cdot \left(Q_n - \frac{n^2}{c} \cdot P_n \right) \cdot z \right\} \cdot \cos(n\theta)\,; \tag{4.29}$$

$$\frac{1}{(c+z)} \cdot \frac{\mathrm{d}\phi}{\mathrm{d}\theta} = -\sum_0^\infty n \cdot \left\{ \frac{P_n}{c} + \left(\frac{Q_n}{c} - \frac{P_n}{c^2} \right) \cdot z \right\} \cdot \sin(n\theta)\,. \tag{4.30}$$

For a point in the ellipse, $z = c\epsilon \cdot \cos(2\theta)$, whence from Equation 4.27, Equation 4.29 and Equation 4.30, we find that the normal velocity of the fluid

$$\begin{aligned} &= \sum_0^\infty \left\{ Q_n \cdot \cos(n\theta) + \frac{\epsilon}{2} \cdot \left[n \cdot (n-2) \cdot \frac{P_n}{c} - Q_n \right] \cdot \cos\left[(n-2) \cdot \theta \right] \right. \\ &\left. + \frac{\epsilon}{2} \cdot \left[n \cdot (n+2) \cdot \frac{P_n}{c} - Q_n \right] \cdot \cos\left[(n+2) \cdot \theta \right] \right\}\,, \end{aligned}$$

which is the same thing as

$$\begin{aligned} &\sum_0^\infty \left\{ \frac{\epsilon}{2} \cdot \left[n \cdot (n-2) \cdot \frac{P_{n-2}}{c} - Q_{n-2} \right] + Q_n \right. \\ &\left. + \frac{\epsilon}{2} \cdot \left[n \cdot (n+2) \cdot \frac{P_{n+2}}{c} - Q_{n+2} \right] \right\} \cdot \cos(n\theta)\,, \end{aligned} \tag{4.31}$$

if we suppose P and Q to be zero when affected with a negative suffix. This expression will have to be equated to the value of $C \cdot \cos(\xi)$ given by Equation 4.26. For the part of the fluid without the radius γ we have[6]

$$\phi = A_0 \cdot \ln(r) + \sum_1^\infty \frac{A_n}{r_n} \cdot \cos(n\theta)\,, \tag{4.31'}$$

since there will be no sines in the expression for ϕ, because the motion is symmetrical with respect to the major axis, and no positive powers of r, because the velocity vanishes at an infinite distance.
From the above value of ϕ we have, for the points at a distance γ from the centre,

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{A_0}{\gamma} - \sum_1^\infty \frac{n \cdot A_n}{\gamma^{n+1}} \cdot \cos(n\theta)\,,$$

$$\frac{1}{r} \cdot \frac{\mathrm{d}\phi}{\mathrm{d}\theta} = -\sum_1^\infty \frac{n \cdot A_n}{\gamma^{n+1}} \cdot \sin(n\theta)\,.$$

Equating the above expressions to the velocities along and perpendicular to the radius vector given by Equation 4.29 and Equation 4.30, when z is put $= k \cdot c$, and then equating coefficients of corresponding sines and cosines, we have

$$(1-k) \cdot Q_n + kn^2 \cdot \frac{P_n}{c} = -\frac{n \cdot A_n}{\gamma^{n+1}}\,, \tag{4.32}$$

$$(1-k) \cdot \frac{P_n}{c} + kQ_n = \frac{A_n}{\gamma^{n+1}}\,, \tag{4.33}$$

when $n > 0$, and equating constant terms we have

$$(1-k) \cdot Q_0 = \frac{A_0}{\gamma}\,,$$

from which equation with Equation 4.32 and Equation 4.33 we have, putting $\gamma = c \cdot (1+k)$,

$$\frac{P_n}{c} = \frac{A_n}{c^{n+1}}\,, \quad Q_n = -\frac{n \cdot A_n}{c^{n+1}}\,, \quad \text{when } \; n > 0\,, \quad \text{and } \; Q_0 = \frac{A_0}{c}\,.$$

[6]The first term of this expression is accurately equal to zero, since there is no expansion or contraction of the solid, (Art. 8., page 29). I have however retained it, in order to render the solution of the problem in the present article independent of the proposition referred to.

Substituting these values in Equation 4.31, it becomes

$$\sum_0^\infty \left\{ \frac{\epsilon}{2} \cdot (n+1) \cdot (n-2) \cdot \frac{A_{n-2}}{c^{n-1}} - \frac{n \cdot A_n}{c^{n+1}} + \frac{\epsilon}{2} \cdot (n+1) \cdot (n+2) \cdot \frac{A_{n+2}}{c^{n+3}} \right\}$$
$$\cdot \cos(n\theta) + \frac{A_0}{c} - \frac{\epsilon \cdot A_0}{2c} \cdot \cos(2\theta) \,.$$

In the case of a circular cylinder the quantities A_0, A_2, A_3, &c. are each zero. In the present case therefore they are small quantities depending on ϵ. Hence, neglecting quantities of the order ϵ^2 in the above expression, it becomes

$$\frac{A_0}{c} + \frac{2\epsilon \cdot A_1}{c^2} \cdot \cos(3\theta) - \sum_1^\infty \frac{n \cdot A_n}{c^{n+1}} \cdot \cos(n\theta) \,,$$

which must be equal to

$$C \cdot \{(1-\epsilon) \cdot \cos(\theta) + \epsilon \cdot \cos(3\theta)\} \,.$$

Equating coefficients of corresponding cosines, we have

$$A_1 = -C \cdot (1-\epsilon) \cdot c^2 \,,$$
$$A_3 = -C\epsilon c^4 \,,$$

and the other quantities A_0, A_2, &c. are of an order higher than ϵ. Hence, for the part of the fluid which lies without the radius γ, we have

$$\phi = -C \cdot \left\{ (1-\epsilon) \cdot \frac{c^2}{r} \cdot \cos(\theta) + \frac{\epsilon c^4}{r^3} \cdot \cos(3\theta) \right\} \,, \tag{4.34}$$

and for the part which lies between that radius and the ellipse we have from Equation 4.28

$$\phi = -Cc \cdot \{(1-\epsilon) \cdot \cos(\theta) + \epsilon \cdot \cos(3\theta)\}$$
$$+ C \cdot \{(1-\epsilon) \cdot \cos(\theta) + 3\epsilon \cdot \cos(3\theta)\} \cdot z - \frac{C}{c} \cdot \cos(\theta) \cdot z^2 \,. \tag{4.35}$$

The value of ϕ given by Equation 4.35 may be deduced from that given by Equation 4.34 by putting $r = c + z$, and expanding as far as to z^2. In the case of the elliptic cylinder then it appears that the same value of ϕ serves for the part of the fluid without, and the part within the radius γ. If the cylinder be moving with a velocity C' in the direction of the minor axis of a section, the value of ϕ will be found from that given by Equation 4.34 by changing the sign of ϵ, putting C' for C, and supposing θ to be measured from the minor axis.
If the cylinder revolve round its axis with an angular velocity ω, the normal velocity

of the surface at any point will be $2\omega\epsilon c \cdot \sin(2\theta)$. Since ϵ^2 is neglected, we may suppose this normal velocity to take place on the surface of a circular cylinder whose radius is c; whence, (Art. 8., page 29), the corresponding value of ϕ will be

$$-\frac{\omega\epsilon c^4}{r^2} \cdot \sin(2\theta)\,.$$

If we suppose all these motions to take place together, we have only, (Art. 5., page 25), to add together the values of ϕ corresponding to each. If we suppose the motion very small, so as to neglect the square of the velocity, we need only retain the terms depending on $\frac{\mathrm{d}\omega}{\mathrm{d}t}$, $\frac{\mathrm{d}C}{\mathrm{d}t}$ and $\frac{\mathrm{d}C'}{\mathrm{d}t}$, in the value of $\frac{\mathrm{d}\phi}{\mathrm{d}t}$, and we may calculate the pressure due to each separately. The resultant of the pressure due to the term $\frac{\mathrm{d}\omega}{\mathrm{d}t}$ will evidently be zero, on account of the symmetry of the corresponding motion, while the resultant couple will be of the order ϵ^2, since the pressure on any point of the surface, and the perpendicular from the centre on the normal at that point, are each of the order ϵ. The pressure due to the term $\frac{\mathrm{d}C}{\mathrm{d}t}$ will evidently have a resultant in the direction of the major axis of a section of the cylinder; and it will be easily proved that the resultant pressure on a length l of the cylinder is $\pi\rho c^2 l \cdot (1-2\epsilon) \cdot \frac{\mathrm{d}C}{\mathrm{d}t}$. That due to the term $\frac{\mathrm{d}C'}{\mathrm{d}t}$ will be $\pi\rho c^2 l \cdot (1+2\epsilon) \cdot \frac{\mathrm{d}C'}{\mathrm{d}t}$, acting along the minor axis. If the cylinder be constrained to oscillate so that its axis oscillates in a direction making an angle α with the major axis, and if C'' be its velocity, which is supposed to be very small, the resultant pressures along the major and minor axes will be

$$\mu \cdot (1-2\epsilon) \cdot \cos(\alpha) \cdot \frac{\mathrm{d}C''}{\mathrm{d}t}$$

and

$$\mu \cdot (1+2\epsilon) \cdot \sin(\alpha) \cdot \frac{\mathrm{d}C''}{\mathrm{d}t}$$

respectively, where μ is the mass of the fluid displaced. Resolving these pressures in the direction of the motion, the resolved part will be

$$\mu \cdot [1 - 2\epsilon \cdot \cos(2\alpha)] \cdot \frac{\mathrm{d}C''}{\mathrm{d}t}$$

or

$$\mu \cdot \left[1 - \frac{e^2}{2} \cdot \cos(2\alpha)\right] \cdot \frac{\mathrm{d}C''}{\mathrm{d}t}\,,$$

e being the eccentricity; so that the effect of the inertia of the fluid will be, to increase the mass of the solid by a mass equal to $\mu \cdot \left[1 - \frac{e^2}{2} \cdot \cos(2\alpha)\right]$, which must be supposed to be collected at the axis.

A similar method of calculation would apply to any given solid differing little either from a circular cylinder or from a sphere. In the latter case it would be necessary to use expansions in series of Laplace's coefficients, instead of expansions in series

of sines and cosines.

4.5. Motion of fluid in a closed box whose interior is of the form of a rectangular parallelepiped

13. The motion being supposed to begin from rest, the motion at any time may be supposed to have been produced by impact (Art. 4., page 22). The motion of the box at any instant may be resolved into a motion of translation and three motions of rotation about three axes parallel to the edges, and passing through the centre of gravity of the fluid, and the part of ϕ due to each of these motions may be calculated separately. Considering any one of the motions of rotation, we shall see that the normal velocity of each face in consequence of it will ultimately be the same as if that face revolved round an axis passing through its centre, and that the latter motion would not alter the volume of the fluid. Consequently, in calculating the part of ϕ due to any one of the angular velocities, we may calculate separately the part due to the motion of each face.
Let the origin be in a corner of the box, the axes coinciding with its edges. Let a, b, c, be these edges, U, V, W, the velocities, parallel to the axes, of the centre of gravity of the interior of the box, ω', ω'', ω''', the angular velocities of the box about axes through this point parallel to those of x, y, z. Let us first consider the part of ϕ due to the motion of the face xz in consequence of the angular velocity ω'''.
The value of ϕ corresponding to this motion must satisfy the equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} = 0\,, \tag{4.36}$$

with the conditions

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = 0\,, \quad \text{when} \;\; x = 0 \text{ or } a\,, \tag{4.37}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = 0\,, \quad \text{when} \;\; y = b\,, \tag{4.38}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = \omega''' \cdot \left(x - \frac{a}{2}\right)\,, \quad \text{when} \;\; y = 0\,, \tag{4.39}$$

within limits corresponding to those of the box.
Now, for a given value of y, the value of ϕ between $x = 0$ and $x = a$ can be expanded in a convergent series of cosines of $\frac{\pi \cdot x}{a}$ and its multiples; and, since Equation 4.37 is satisfied, the series by which $\frac{\mathrm{d}\phi}{\mathrm{d}x}$ will be expressed will also hold good for the limiting values of x, and will be convergent. The general value of ϕ then will be of the form $\sum_0^\infty Y_n \cdot \cos\left(\frac{n\pi x}{a}\right)$. Substituting in Equation 4.36, and equating coefficients of corresponding cosines, which may be done, since any function of x

can be expanded in but one such series of cosines between the limits 0 and a, we find that the general value of Y_n is $C \cdot e^{\frac{n\pi y}{a}} + C' \cdot e^{-\frac{n\pi y}{a}}$, or, changing the constants,

$$Y_n = A_n \cdot \left\{ e^{\frac{n\pi \cdot (b-y)}{a}} + e^{-\frac{n\pi \cdot (b-y)}{a}} \right\} + B_n \cdot \left\{ e^{\frac{n\pi y}{a}} + e^{-\frac{n\pi y}{a}} \right\} ,$$

when $n > 0$, and for $n = 0$,

$$Y_0 = A_0 \cdot y + B_0 .$$

From the condition according to Equation 4.38 we have

$$A_0 + \frac{\pi}{a} \cdot \sum_1^\infty n B_n \cdot \left\{ e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}} \right\} \cdot \cos\left(\frac{n\pi x}{a}\right) = 0 :$$

whence $A_0 = 0$, $B_n = 0$, and, omitting B_0,

$$\phi = \sum_1^\infty A_n \cdot \left\{ e^{\frac{n\pi \cdot (b-y)}{a}} + e^{-\frac{n\pi \cdot (b-y)}{a}} \right\} \cdot \cos\left(\frac{n\pi x}{a}\right) .$$

From the condition according to Equation 4.39, we have

$$-\frac{\pi}{a} \cdot \sum_1^\infty n A_n \left\{ e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}} \right\} \cdot \cos\left(\frac{n\pi x}{a}\right) = \omega''' \cdot \left(x - \frac{a}{2}\right) .$$

Determining the coefficients in the usual manner, we have

$$A_n = \frac{2a^2\omega'''}{n^3\pi^3} \cdot \{1 - (-1)^n\} \cdot \frac{1}{e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}} ;$$

whence

$$\phi = \frac{4a^2\omega'''}{\pi^3} \cdot \sum_0 \frac{1}{n^3} \cdot \frac{e^{\frac{n\pi \cdot (b-y)}{a}} + e^{-\frac{n\pi \cdot (b-y)}{a}}}{e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}} \cdot \cos\left(\frac{n\pi x}{a}\right) ,$$

putting $\sum_0$, for shortness, to denote the sum corresponding to *odd* integral values of n from 1 to ∞.

It is evident that the value of ϕ corresponding to the motion of the opposite face in consequence of the angular velocity ω''' will be found from that just given by putting $(b - y)$ for y, and changing the sign of ω'''; whence the value corresponding to the motion of these two faces in consequence of ω''' will be

$$\frac{4a^2\omega'''}{\pi^3} \cdot \sum_0 \frac{1}{n^3} \cdot \frac{\left\{ e^{\frac{n\pi b}{a}} - 1 \right\} \cdot e^{-\frac{n\pi y}{a}} + \left\{ e^{-\frac{n\pi b}{a}} - 1 \right\} \cdot e^{\frac{n\pi y}{a}}}{e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}} \cdot \cos\left(\frac{n\pi x}{a}\right) .$$

Let this expression be denoted by $\omega'''\psi(x, a, y, b)$. It is evident that the part of ϕ due to the motion of the two faces parallel to the plane yz will be got by interchanging x

and y, a and b, and changing the sign of ω''' in the last expression, and will therefore be $-\omega'''\psi(y,b,x,a)$. The parts of ϕ corresponding to the angular velocities ω', ω'', will be got by interchanging the requisite quantities. Also the part of ϕ due to the velocities U, V, W, will be $Ux+Vy+Wz$, (Art. 7., page 27), and therefore we have for the complete value of ϕ

$$\begin{aligned} Ux+Vy+Wz &+\omega'''\cdot\{\psi(x,a,y,b)-\psi(y,b,x,a)\} \\ &+\omega'\cdot\{\psi(y,b,z,c)-\psi(z,c,y,b)\} \\ &+\omega''\cdot\{\psi(z,c,x,a)-\psi(x,a,z,c)\}\ . \end{aligned}$$

According to Art. 7., page 27 we may consider separately the motion of translation of the box and fluid, and the motion of rotation about the centre of gravity of the latter; and the whole pressure will be compounded of the pressures due to each. The pressures at the several points of the box due to the motion of translation will have a single resultant, which will be the same as if the mass of the fluid were collected at its centre of gravity. Those due to the motion of rotation will have a single resultant couple, to calculate which we have

$$\phi=\omega'''\cdot\{\psi(x,a,y,b)-\psi(y,b,x,a)\}+\ \&\text{c.}$$

Since for the motion of rotation there is no resultant force, we may find the resultant couple of the pressures round *any* origin, that for instance which has been chosen. If now we suppose the motion very small, so as to neglect the square of the velocity, we may find $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ as if the axes were fixed in space. We have then for the motion of rotation

$$p=-\rho\cdot\frac{\mathrm{d}\omega'''}{\mathrm{d}t}\cdot\{\psi(x,a,y,b)-\psi(y,b,x,a)\}-\ \&\text{c.}$$

Hence we may calculate separately the couples due to each of the quantities $\frac{\mathrm{d}\omega'''}{\mathrm{d}t}$, $\frac{\mathrm{d}\omega'}{\mathrm{d}t}$ and $\frac{\mathrm{d}\omega''}{\mathrm{d}t}$. It is evident from the symmetry of the motion that this due to $\frac{\mathrm{d}\omega'''}{\mathrm{d}t}$ will act round the axis of z, and that the pressures on the two faces perpendicular to that axis will have resultants which are equal and opposite. Also, since $\psi(a,a,y,b)=-\psi(0,a,y,b)$ and $\psi(x,a,b,b)=-\psi(x,a,0,b)$, it will be seen that the couples due to the pressures on the faces perpendicular to the axes of x and y will be twice as great respectively as those due to the pressures on the planes yz and xz. The pressure on the element $\mathrm{d}y\,\mathrm{d}z$ of the plane yz will be $p_{x=0}\,\mathrm{d}y\,\mathrm{d}z$, and the moment of this pressure round the axis of z, reckoned positive when it tends to turn the box from x to y, will be

$$-\rho\cdot\frac{\mathrm{d}\omega'''}{\mathrm{d}t}\cdot y\,\{\psi(0,a,y,b)-\psi(y,b,0,a)\}\,\mathrm{d}y\,\mathrm{d}z\,.$$

Substituting the values of the functions, integrating from $y=0$ to $y=b$, and from $z=0$ to $z=c$, replacing $\sum_0\frac{1}{n^5}$ by its value $\frac{\pi^4}{96}$, and reducing the other terms, it

will be found that the couple due to the pressure on the plane yz is

$$\frac{\rho a^3 bc}{24} \cdot \frac{\mathrm{d}\omega'''}{\mathrm{d}t} - \frac{8\rho a^4 c}{\pi^5} \cdot \frac{\mathrm{d}\omega'''}{\mathrm{d}t} \cdot \sum_0 \frac{1}{n^5} \cdot \frac{1 - e^{-\frac{n\pi b}{a}}}{1 + e^{-\frac{n\pi b}{a}}} - \frac{8\rho b^4 c}{\pi^5} \cdot \frac{\mathrm{d}\omega'''}{\mathrm{d}t} \cdot \sum_0 \frac{1}{n^5} \cdot \frac{1 - e^{-\frac{n\pi a}{b}}}{1 + e^{-\frac{n\pi a}{b}}} \cdot$$

We shall get the couple due to the pressure on the plane xz by interchanging a and b, changing the sign of ω''', and measuring the couple in the opposite direction, or, which is the same, by merely interchanging a and b. Adding together these two couples and doubling their sum we shall find that the couple due to $\frac{\mathrm{d}\omega'''}{\mathrm{d}t}$ is $-C \cdot \frac{\mathrm{d}\omega'''}{\mathrm{d}t}$, where

$$C = \frac{32\rho c}{\pi^5} \cdot \sum_0 \frac{1}{n^5} \cdot \left\{ a^4 \cdot \frac{1 - e^{-\frac{n\pi b}{a}}}{1 + e^{-\frac{n\pi b}{a}}} + b^4 \cdot \frac{1 - e^{-\frac{n\pi a}{b}}}{1 + e^{-\frac{n\pi a}{b}}} \right\} - \frac{\rho abc}{12} \cdot \left(a^2 + b^2\right) . \quad (4.40)$$

Similarly, the couple due to $\frac{\mathrm{d}\omega'}{\mathrm{d}t}$ will be $-A \cdot \frac{\mathrm{d}\omega'}{\mathrm{d}t}$, tending to turn the box from y to z, and that due to $\frac{\mathrm{d}\omega''}{\mathrm{d}t}$ will be $-B \cdot \frac{\mathrm{d}\omega''}{\mathrm{d}t}$, tending to turn the box from z to x, where A and B are derived from C by interchanging the requisite quantities. Hence, considering the motions both of translation and rotation of the box, we see that the small motions of the box will take place as if the fluid were replaced by a solid having the same mass, centre of gravity, and principal axes, and having A, B and C for its principal moments. This will be true whether forces act on the fluid or not, provided that if there are any they are of the kind mentioned in Art. 1., page 17.

Putting $A_\prime$, $B_\prime$, $C_\prime$, for the principal moments of inertia of the solidified fluid we have

$$C_\prime = \frac{\rho abc}{12} \cdot \left(a^2 + b^2\right) .$$

Taking the ratio of C to $C_\prime$, replacing each term such as

$$\frac{1 - e^{-\frac{n\pi b}{a}}}{1 + e^{-\frac{n\pi b}{a}}} \quad \text{by} \quad 1 - \frac{2 \cdot e^{-\frac{n\pi b}{a}}}{1 + e^{-\frac{n\pi b}{a}}} ,$$

putting for

$$\frac{384}{\pi^5} \cdot \sum_0 \frac{1}{n^5}$$

its approximate value 1.260497, and for $\frac{384}{\pi^5}$ its approximate value 1.254821, and employing subsidiary angles, we have[7][8]

$$\frac{C}{C_{\prime}} = 1.260497 \cdot \frac{a^4+b^4}{ab\cdot(a^2+b^2)} - 1.254821 \cdot \left\{ \frac{a^3}{b\cdot(a^2+b^2)} \sum_0 \frac{1}{n^5} \cdot \operatorname{versin}(2\theta_n) + \frac{b^3}{a\cdot(a^2+b^2)} \sum_0 \frac{1}{n^5} \cdot \operatorname{versin}(2\theta'_n) \right\} - 1\,,$$

where

$$\tan(\theta_n) = e^{-\frac{n\pi b}{2a}}\,,$$
$$\tan(\theta'_n) = e^{-\frac{n\pi a}{2b}}\,,$$

so that

$$L\cdot\tan(\theta_n) = 10 - k\cdot\frac{nb}{a}\,,$$
$$L\cdot\tan(\theta'_n) = 10 - k\cdot\frac{na}{b}\,,$$

where $k = 0.6821882$.

The numerical calculation of this ratio is very easy, on account of the great rapidity with which the series contained in it converges, both on account of the coefficients, and on account of the rapid diminution of the angles θ_n and θ'_n. The values of $\frac{A}{A_{\prime}}$ and $\frac{B}{B_{\prime}}$ will be derived from that of $\frac{C}{C_{\prime}}$ by putting c for a in the first case, and c for b in the second. The calculation of the small motions of the box will thus be reduced to a question of ordinary rigid dynamics.

These results appear capable of being accurately compared with experiment. For this purpose it will only be necessary to attach a box, capable of containing fluid, to a rigid body oscillating as a pendulum. The box may itself form the rigid body. The centre of gravity of the interior of the box should be in a vertical plane passing through the axis of suspension, which will be known by observing whether the position of equilibrium of the whole is affected by filling the box with fluid. The mass, moment of inertia, and depth of the centre of gravity of the solid, including the box, must first be found. The last of these may be found by loading the upper part of the oscillating body till the equilibrium just becomes unstable: the moment

[7] The versine or versedsine is a trigonometric function found in some of the earliest (Sanskrit Aryabhatia) trigonometric tables. The versine of an angle is 1 minus its cosine. Expressed in terms of common trigonometric functions sine, cosine, and tangent, the versine is equal to $\operatorname{versin}\theta = 1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right) = \sin\theta\,\tan\left(\frac{\theta}{2}\right)$.

[8] It will be shewn further on, in a supplement to this paper, that either of these two infinite series may be expressed by means of the other, so that we shall have only one of the infinite series to calculate in any case, for which we may choose the more rapidly convergent.

of inertia will then be found by means of the time of oscillation when the weight is removed; or else both may be determined by the times of oscillation when the solid is loaded with another of known mass and form and placed in a known position, and again when it is not loaded. The same must then be done when the box is filled with fluid. We shall thus determine the moment of inertia and depth of the centre of gravity of the fluid; and, subtracting the moment of inertia due to the motion of translation of the fluid, we shall thus get that due to the motion of rotation of the box, and thus determine in succession by observation the quantities A, B and C, or any one of them. These quantities might also be determined by making the box oscillate by torsion, and observing the time of oscillation. It must be remembered that the moment of inertia due to the motion of translation of the centre of gravity of the fluid, being capable of being derived from the general dynamical principle, that the motion of the centre of gravity of any system is the same as if the whole mass were collected there, and the external moving forces applied there, is of no use whatever in determining the question of the equality of the pressure in all directions, or that of the amount of friction. It would seem to be most convenient to have the centre of gravity of the fluid in the axis of suspension. In this case if M, M', be the masses of the solid and fluid, μ, μ', their moments of inertia, t, t', the times of oscillation, in seconds, when the box is empty and when it is full respectively, h the depth of the centre of gravity of the solid, l the length of the second's pendulum, we have

$$\mu = lt^2 Mh\,,$$
$$\mu + \mu' = lt'^2 Mh\,;$$

whence

$$\mu' = l \cdot (t'^2 - t^2) \cdot Mh\,.$$

If the centre of gravity of the fluid be at a depth h' below the axis of suspension, we shall have

$$\mu' = l \cdot (t'^2 - t^2) \cdot Mh + lt'^2 M'h'\,;$$

in this case $\mu' - M'h'^2$ will be the moment of inertia due to the motion of rotation of the box.

When one of the quantities a, b, becomes infinitely great compared with the other, the ratio $\frac{C}{C_l}$ becomes 1, as will be seen from Equation 4.40. This result might have been expected. When $a = b$, the value of $\frac{C}{C_l}$ is 0.156537.

The experiment of the box appears capable of great variety as well as accuracy. We may take boxes in which the edges have various ratios to each other, and may make the same box oscillate in various positions.

4.6. Initial motion in a rectangular box, the several points of the surface of which are moved with given velocities, consistent with the condition that the volume of the fluid is not altered

15. Employing the same notation as in the last case, let $F(x, y)$ be the given normal velocity at any point of the face in the plane xy. Let $\int_0^a \int_0^b F(x, y)\,\mathrm{d}x\,\mathrm{d}y = Wab$, and let

$$F(x, y) = f(x, y) + W \ :$$

then, since the normal motion of the above face due to the function $f(x, y)$ does not alter the volume of the fluid, we may consider separately the part of ϕ due to this quantity. For this part we have

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} + \frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} = 0\,, \tag{4.41}$$

with the conditions

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = 0\,, \quad \text{when} \quad x = 0 \quad \text{or } a\,, \tag{4.42}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = 0\,, \quad \text{when} \quad y = 0 \quad \text{or } b\,, \tag{4.43}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = 0\,, \quad \text{when} \quad z = c\,, \tag{4.44}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = f(x, y)\,, \quad \text{when} \quad z = 0\,, \tag{4.45}$$

within limits corresponding to those of the box.
For a given value of z the value of ϕ from $x = 0$ to $x = a$ and from $y = 0$ to $y = b$ may be expanded in a series of the form

$$\sum_0^\infty \sum_0^\infty{}' P_{m,n} \cdot \cos\left(\frac{m\pi x}{a}\right) \cdot \cos\left(\frac{n\pi y}{b}\right)\,,$$

the sign $\sum$ referring to m, and $\sum'$ to n: and since the values of ϕ, $\frac{\mathrm{d}\phi}{\mathrm{d}x}$ and $\frac{\mathrm{d}\phi}{\mathrm{d}y}$ do not alter abruptly, and Equation 4.42 and Equation 4.43 are satisfied, it follows that the series by which ϕ, $\frac{\mathrm{d}\phi}{\mathrm{d}x}$ and $\frac{\mathrm{d}\phi}{\mathrm{d}y}$ are expressed are convergent, and hold good for the limiting values of x and y. Substituting the value of ϕ just given in Equation 4.41, equating to zero coefficients of corresponding cosines, and introducing the condition

according to Equation 4.44, we have, omitting the constant, or supposing $A_{0,0} = 0$,

$$\phi = \sum_0^\infty \sum_0^\infty{}' A_{m,n} \cdot \left\{ e^{\frac{p\pi\cdot(c-z)}{c}} + e^{-\frac{p\pi\cdot(c-z)}{c}} \right\} \cdot \cos\left(\frac{m\pi x}{a}\right) \cdot \cos\left(\frac{n\pi y}{b}\right) ,$$

where

$$\frac{p^2}{c^2} = \frac{m^2}{a^2} + \frac{n^2}{b^2} .$$

Determining the coefficients such as $A_{m,n}$ from the condition according to Equation 4.45 in the usual manner we have, m and n being > 0,[9]

$$A_{m,n} = -\frac{4c}{\pi pab} \cdot \frac{1}{(e^{p\pi} - e^{-p\pi})} \cdot \int_0^a \int_0^b f(x,y) \cdot \cos\left(\frac{m\pi x}{a}\right) \cdot \cos\left(\frac{n\pi y}{b}\right) \mathrm{d}x\, \mathrm{d}y ,$$

$$A_{0,n} = -\frac{2}{\pi na} \cdot \frac{1}{\left(e^{\frac{n\pi c}{b}} - e^{-\frac{n\pi c}{b}}\right)} \cdot \int_0^a \int_0^b f(x,y) \cdot \cos\left(\frac{n\pi y}{b}\right) \mathrm{d}x\, \mathrm{d}y ,$$

with a similar expression for $A_{m,0}$, whence the value of ϕ corresponding to $f(x,y)$ is known. In a similar manner we may find the values corresponding to the similar functions belonging to each of the other faces. If W' be the quantity corresponding to W for the face opposite to the plane xy, and U, U', correspond to W, W', for the faces perpendicular to the axis of x, and if V, V', be the corresponding quantities for y, there remains only to be found the part of ϕ due to these six quantities. Since U, U', are the velocities parallel to the axis of x of the faces perpendicular to that axis, and so for V, V', &c., the motion corresponding to these six quantities may be resolved into three motions of translation parallel to the three axes, the velocities being U, V and W, and that motion which is due to the motions of the faces opposite to the planes yz, xz, xy, moving with velocities $U' - U$, $V' - V$, $W' - W$, parallel to the axes of x, y, z, respectively. The condition that the volume of the fluid remains the same requires that

$$\frac{1}{a} \cdot (U' - U) + \frac{1}{b} \cdot (V' - V) + \frac{1}{c} \cdot (W' - W) = 0 .$$

It will be found that the velocities

$$u = \frac{x}{a} \cdot (U' - U) ,$$
$$v = \frac{y}{b} \cdot (V' - V) ,$$
$$w = \frac{z}{c} \cdot (W' - W) ,$$

[9]The function $f(x,y)$ in these integrals may be replaced by $F(x,y)$, since $\int_0^a \int_0^b W \cdot \cos\left(\frac{n\pi y}{b}\right) \cdot \cos\left(\frac{n\pi x}{a}\right) \mathrm{d}x\, \mathrm{d}y = 0$, unless $m = n = 0$.

satisfy all the requisite conditions. Hence the part of ϕ due to the six quantities U, U', V, V', W, W', is

$$Ux + Vy + Wz + (U' - U) \cdot \frac{x^2}{2a} + (V' - V) \cdot \frac{y^2}{2b} + (W' - W) \cdot \frac{z^2}{2c}.$$

This quantity, added to the six others which have already been given, gives the value of ϕ which contains the complete solution of the problem.
The case of motion which has just been given seems at first sight to be an imaginary one, capable of no practical application. It may however be applied to the determination of the small motion of a ball pendulum oscillating in a case in the form of a rectangular parallelepiped, the dimensions of the case being great compared with the radius of the ball. For this purpose it will be necessary to calculate the motion of the ball reflected from the case, by means of the formulæ just given, and then the motion again reflected from the sphere, exactly as has been done in the case of a rigid plane (Art. 10, page 41). In the present instance however the result contains definite integrals, the numerical calculation of which would be very troublesome.

Part II.

On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids

1. Introductory Notes

The equations of Fluid Motion commonly employed depend upon the fundamental hypothesis that the mutual action of two adjacent elements of the fluid is normal to the surface which separates them. From this assumption the equality of pressure in all directions is easily deduced, and then the equations of motion are formed according to D'Alembert's principle. This appears to me the most natural light in which to view the subject; for the two principles of the absence of tangential action, and of the equality of pressure in all directions ought not to be assumed as independent hypotheses, as is sometimes done, inasmuch as the latter is a necessary consequence of the former[1]. The equations of motion so formed are very complicated, but yet they admit of solution in some instances, especially in the case of small oscillations. The results of the theory agree on the whole with observation, so far as the time of oscillation is concerned. But there is a whole class of motions of which the common theory takes no cognizance whatever, namely, those which depend on the tangential action called into play by the sliding of one portion of a fluid along another, or of a fluid along the surface of a solid, or of a different fluid, that action in fact which performs the same part with fluids that friction does with solids.

Thus, when a ball pendulum oscillates in an indefinitely extended fluid, the common theory gives the arc of oscillation constant. Observation however shews that it diminishes very rapidly in the case of a liquid, and diminishes, but less rapidly, in the case of an elastic fluid. It has indeed been attempted to explain this diminution by supposing a friction to act on the ball, and this hypothesis may be approximately true, but the imperfection of the theory is shewn from the circumstance that no account is taken of the equal and opposite friction of the ball on the fluid.

Again, suppose that water is flowing down a straight aqueduct of uniform slope, what will be the discharge corresponding to a given slope, and a given form of the bed? Of what magnitude must an aqueduct be, in order to supply a given place with a given quantity of water? Of what form must it be, in order to ensure a given supply of water with the least expense of materials in the construction? These, and similar questions are wholly out of the reach of the common theory of Fluid Motion, since they entirely depend on the laws of the transmission of that tangential action which in it is wholly neglected. In fact, according to the common theory the water ought to flow on with uniformly accelerated velocity; for even the supposition of a

[1]This may be easily shewn by the consideration of a tetrahedron of the fluid, as in Art. 4., page 82

certain friction against the bed would be of no avail, for such friction could not be transmitted through the mass. The practical importance of such questions as those above mentioned has made them the object of numerous experiments, from which empirical formulæ have been constructed. But such formulæ, although fulfilling well enough the purposes for which they were constructed, can hardly be considered as affording us any material insight into the laws of nature; nor will they enable us to pass from the consideration of the phenomena from which they were derived to that of others of a different class, although depending on the same causes.

In reflecting on the principles according to which the motion of a fluid ought to be calculated when account is taken of the tangential force, and consequently the pressure not supposed the same in all directions, I was led to construct the theory explained in the first section of this paper, or at least the main part of it, which consists of Equation 2.13, and of the principles on which they are formed. I afterwards found that POISSON had written a memoir on the same subject, and on referring to it I found that he had arrived at the same equations. The method which he employed was however so different from mine that I feel justified in laying the latter before this Society[2]. The leading principles of my theory will be found in the hypotheses of Art. 1., page 73, and in Art. 3., page 78.

The second section forms a digression from the main object of this paper, and at first sight may appear to have little connexion with it. In this section I have, I think, succeeded in shewing that LAGRANGE's proof of an important theorem in the ordinary theory of Hydrodynamics is untenable. The theorem to which I refer is the one of which the object is to shew that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$, (using the common notation,) is always an exact differential when it is so at one instant. I have mentioned the principles of M. CAUCHY's proof, a proof, I think, liable to no sort of objection. I have also given a new proof of the theorem, which would have served to establish it had M. CAUCHY not been so fortunate as to obtain three first integrals of the general equations of motion. As it is, this proof may possibly be not altogether useless.

POISSON, in the memoir to which I have referred, begins with establishing, according to his theory, the equations of equilibrium and motion of elastic solids, and makes the equations of motion of fluids depend on this theory. On reading his memoir, I was led to apply to the theory of elastic solids principles precisely analogous to those which I have employed in the case of fluids. The formation of the equations, according to these principles, forms the subject of chapter 4.

The equations at which I have thus arrived contain two arbitrary constants, whereas POISSON's equations contain but one. In chapter 5 I have explained the principles of POISSON's theories of elastic solids, and of the motion of fluids, and pointed out what appear to me serious objections against the truth of one of the hypotheses

[2]The same equations have also been obtained by NAVIER in the case of an incompressible fluid, (*Mémoires de l'Académie*, T. VI., `https://www.biodiversitylibrary.org/page/16364086`) but his principles differ from mine still more than do POISSON's.

which he employs in the former. This theory seems to be very generally received, and in consequence it is usual to deduce the measure of the cubical compressibility of elastic solids from that of their extensibility, when formed into rods or wires, or from some quantity of the same nature. If the views which I have explained in this section be correct, the cubical compressibility deduced in this manner is too great, much too great in the case of the softer substances, and even the softer metals. The equations of chapter 4 have, I find, been already obtained by M. Cauchy in his *Exercises Mathématiques*, except that he has not considered the effect of the heat developed by sudden compression. The method which I have employed is different from his, although in some respects it much resembles it.
The equations of motion of elastic solids given in chapter 4 are the same as those to which different authors have been led, as being the equations of motion of the luminiferous ether in vacuum. It may seem strange that the same equations should have been arrived at for cases so different; and I believe this has appeared to some a serious objection to the employment of those equations in the case of light. I think the reflections which I have made at the end of chapter 5, where I have examined the consequences of the law of continuity, a law which seems to pervade nature, may tend to remove the difficulty.

2. Explanation of the Theory of Fluid Motion proposed. Formation of the Differential Equations. Application of these Equations to a few simple cases

1. Before entering on the explanation of this theory, it will be necessary to define, or fix the precise meaning of a few terms which I shall have occasion to employ.
In the first place, the expression "the velocity of a fluid at any particular point" will require some notice. If we suppose a fluid to be made up of ultimate molecules, it is easy to see that these molecules must, in general, move among one another in an irregular manner, through spaces comparable with the distances between them, when the fluid is in motion. But since there is no doubt that the distance between two adjacent molecules is quite insensible, we may neglect the irregular part of the velocity, compared with the common velocity with which all the molecules in the neighbourhood of the one considered are moving. Or, we may consider the mean velocity of the molecules in the neighbourhood of the one considered, apart from the velocity due to the irregular motion. It is this regular velocity which I shall understand by the *velocity of a fluid at any point*, and I shall accordingly regard it as varying continuously with the co-ordinates of the point.
Let P be any material point in the fluid, and consider the instantaneous motion of a very small element E of the fluid about P. This motion is compounded of a motion of translation, the same as that of P, and of the motion of the several points of E relatively to P. If we conceive a velocity equal and opposite to that of P impressed on the whole element, the remaining velocities form what I shall call the *relative velocities* of the points of the fluid about P; and the motion expressed by these velocities is what I shall call the *relative motion* in the neighbourhood of P.
It is an undoubted result of observation that the molecular forces, whether in solids, liquids, or gases, are forces of enormous intensity, but which are sensible at only insensible distances. Let E' be a very small element of the fluid circumscribing E, and of a thickness greater than the distance to which the molecular forces are sensible. The forces acting on the element E are the external forces, and the

pressures arising from the molecular action of E'. If the molecules of E were in positions in which they could remain at rest if E were acted on by no external force and the molecules of E' were held in their actual positions, they would be in what I shall call a state of *relative equilibrium*. Of course they may be far from being in a state of actual equilibrium. Thus, an element of fluid at the top of a wave may be sensibly in a state of relative equilibrium, although far removed from its position of equilibrium. Now, in consequence of the intensity of the molecular forces, the pressures arising from the molecular action on E will be very great compared with the external moving forces acting on E. Consequently the state of relative equilibrium, or of relative motion, of the molecules of E will not be sensibly affected by the external forces acting on E. But the pressures in different directions about the point P depend on that state of relative equilibrium or motion, and consequently will not be sensibly affected by the external moving forces acting on E. For the same reason they will not be sensibly affected by any motion of rotation common to all the points of E; and it is a direct consequence of the second law of motion, that they will not be affected by any motion of translation common to the whole element. If the molecules of E were in a state of relative equilibrium, the pressure would be equal in all directions about P, as in the case of fluids at rest. Hence I shall assume the following principle:–

That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned.

Let us see how far this principle will lead us when it is carried out.

2. It will be necessary now to examine the nature of the most general instantaneous motion of an element of a fluid. The proposition in this article is however purely geometrical, and may be thus enunciated:– "Supposing space, or any portion of space, to be filled with an infinite number of points which move in any continuous manner, retaining their identity, to examine the nature of the instantaneous motion of any elementary portion of these points."
Let u, v, w be the resolved parts, parallel to the rectangular axes Ox, Oy, Oz, of the velocity of the point P, whose co-ordinates at the instant considered are x, y, z. Then the relative velocities at the point P', whose co-ordinates are $x + x'$, $y + y'$, $z + z'$, will be

$$\frac{\mathrm{d}u}{\mathrm{d}x}x' + \frac{\mathrm{d}u}{\mathrm{d}y}y' + \frac{\mathrm{d}u}{\mathrm{d}z}z' \quad \text{parallel to } x\,,$$

$$\frac{\mathrm{d}v}{\mathrm{d}x}x' + \frac{\mathrm{d}v}{\mathrm{d}y}y' + \frac{\mathrm{d}v}{\mathrm{d}z}z' \quad \text{parallel to } y\,,$$
$$\frac{\mathrm{d}w}{\mathrm{d}x}x' + \frac{\mathrm{d}w}{\mathrm{d}y}y' + \frac{\mathrm{d}w}{\mathrm{d}z}z' \quad \text{parallel to } z\,,$$

neglecting squares and products of x', y', z'. Let these velocities be compounded of those due to the angular velocities ω', ω'', ω''' about the axes of x, y, z, and of the velocities U, V, W parallel to x, y, z. The linear velocities due to the angular velocities being $\omega''z' - \omega'''y$, $\omega'''x' - \omega'z'$, $\omega'y' - \omega''x'$ parallel to the axes of x, y, z, we shall therefore have

$$U = \frac{\mathrm{d}u}{\mathrm{d}x}x' + \left(\frac{\mathrm{d}u}{\mathrm{d}y} + \omega'''\right)y' + \left(\frac{\mathrm{d}u}{\mathrm{d}z} - \omega''\right)z'\,,$$
$$V = \left(\frac{\mathrm{d}v}{\mathrm{d}x} - \omega'''\right)x' + \frac{\mathrm{d}v}{\mathrm{d}y}y' + \left(\frac{\mathrm{d}v}{\mathrm{d}z} + \omega'\right)z'\,,$$
$$W = \left(\frac{\mathrm{d}w}{\mathrm{d}x} + \omega''\right)x' + \left(\frac{\mathrm{d}w}{\mathrm{d}y} - \omega'\right)y' + \frac{\mathrm{d}w}{\mathrm{d}z}z'\,.$$

Since ω', ω'', ω''' are arbitrary, let them be so assumed that

$$\frac{\mathrm{d}U}{\mathrm{d}y'} = \frac{\mathrm{d}V}{\mathrm{d}x'}\,, \quad \frac{\mathrm{d}V}{\mathrm{d}z'} = \frac{\mathrm{d}W}{\mathrm{d}y'}\,, \quad \frac{\mathrm{d}W}{\mathrm{d}x'} = \frac{\mathrm{d}U}{\mathrm{d}z'}\,,$$

which gives

$$\left.\begin{aligned}
\omega' &= \frac{1}{2}\cdot\left(\frac{\mathrm{d}w}{\mathrm{d}y} - \frac{\mathrm{d}v}{\mathrm{d}z}\right)\,,\\
\omega'' &= \frac{1}{2}\cdot\left(\frac{\mathrm{d}u}{\mathrm{d}z} - \frac{\mathrm{d}w}{\mathrm{d}x}\right)\,,\\
\omega''' &= \frac{1}{2}\cdot\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}y}\right)\,,
\end{aligned}\right\} \tag{2.1}$$

$$\left.\begin{aligned}
U &= \frac{\mathrm{d}u}{\mathrm{d}x}x' + \frac{1}{2}\cdot\left(\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}x}\right)y' + \frac{1}{2}\cdot\left(\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}x}\right)z'\,,\\
V &= \frac{1}{2}\cdot\left(\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}y}\right)x' + \frac{\mathrm{d}v}{\mathrm{d}y}y' + \frac{1}{2}\cdot\left(\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y}\right)z'\,,\\
W &= \frac{1}{2}\cdot\left(\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z}\right)x' + \frac{1}{2}\cdot\left(\frac{\mathrm{d}w}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}z}\right)y' + \frac{\mathrm{d}w}{\mathrm{d}z}z'\,.
\end{aligned}\right\} \tag{2.2}$$

The quantities ω', ω'', ω''' are what I shall call the *angular velocities of the fluid* at the point considered. This is evidently an allowable definition, since, in the particular case in which the element considered moves as a solid might do, these

quantities coincide with the angular velocities considered in rigid dynamics. A further reason for this definition will appear in chapter 4.
Let us now investigate whether it is possible to determine x', y', z' so that, considering only the relative velocities U, V, W, the line joining the points P, P' shall have no angular motion. The conditions to be satisfied, in order that this may be the case, are evidently that the increments of the relative co-ordinates x', y', z' of the second point shall be ultimately proportional to those co-ordinates. If e be the rate of extension of the line joining the two points considered, we shall therefore have

$$\left.\begin{aligned} F\cdot x' + h\cdot y' + g\cdot z' &= e\cdot x' \,, \\ h\cdot x' + G\cdot y' + f\cdot z' &= e\cdot y' \,, \\ g\cdot x' + f\cdot y' + H\cdot z' &= e\cdot z' \,; \end{aligned}\right\} \tag{2.3}$$

where

$$\begin{aligned} F &= \frac{\mathrm{d}u}{\mathrm{d}x} \,, \\ G &= \frac{\mathrm{d}v}{\mathrm{d}y} \,, \\ H &= \frac{\mathrm{d}w}{\mathrm{d}z} \,, \\ 2f &= \frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y} \,, \\ 2g &= \frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z} \,, \\ 2h &= \frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}x} \,. \end{aligned}$$

If we eliminate from Equation 2.3 the two ratios which exist between the three quantities x', y', z', we get the well-known cubic equation

$$(e-F)\cdot(e-G)\cdot(e-H) - f^2\cdot(e-F) - g^2\cdot(e-G) - h^2\cdot(e-H) - 2fgh = 0 \,, \tag{2.4}$$

which occurs in the investigation of the principal axes of a rigid body, and in various others. As in these investigations, it may be shewn that there are in general three directions, at right angles to each other, in which the point P' may be situated so as to satisfy the required conditions. If two of the roots of Equation 2.4 are equal, there is one such direction corresponding to the third root, and an infinite number of others situated in a plane perpendicular to the former; and if the three roots of Equation 2.4 are equal, a line drawn in any direction will satisfy the required conditions.

The three directions which have just been determined I shall call *axes of extension.*

They will in general vary from one point to another, and from one instant of time to another. If we denote the three roots of Equation 2.4 by e', e'', e''', and if we take new rectangular axes $Ox_\prime$, $Oy_\prime$, $Oz_\prime$, parallel to the axes of extension, and denote by $u_\prime$, $U_\prime$, &c. the quantities referred to these axes corresponding to u, U, &c., Equation 2.3 must be satisfied by $y'_\prime = 0$, $z'_\prime = 0$, $e = e'$, by $x'_\prime = 0$, $z'_\prime = 0$, $e = e''$, and by $x'_\prime = 0$, $y'_\prime = 0$, $e = e'''$, which requires that $f_\prime = 0$, $g_\prime = 0$, $h_\prime = 0$, and we have

$$e' = F_\prime = \frac{\mathrm{d}u_\prime}{\mathrm{d}x_\prime},$$
$$e'' = G_\prime = \frac{\mathrm{d}v_\prime}{\mathrm{d}y_\prime},$$
$$e''' = H_\prime = \frac{\mathrm{d}w_\prime}{\mathrm{d}z_\prime}.$$

The values of $U_\prime$, $V_\prime$, $W_\prime$, which correspond to the residual motion after the elimination of the motion of rotation corresponding to ω', ω'' and ω''', are

$$U_\prime = e' \cdot x'_\prime,$$
$$V_\prime = e'' \cdot y'_\prime,$$
$$W_\prime = e''' \cdot z'_\prime.$$

The angular velocity of which ω', ω'' and ω''' are the components is independent of the arbitrary directions of the co-ordinate axes: the same is true of the directions of the axes of extension, and of the values of the roots of Equation 2.4. This might be proved in various ways; perhaps the following is the simplest. The conditions by which ω', ω'' and ω''' are determined are those which express that the relative velocities U, V, W, which remain after eliminating a certain angular velocity, are such that $U\,\mathrm{d}x' + V\,\mathrm{d}y' + W\,\mathrm{d}z'$ is ultimately an exact differential, that is to say when squares and products of x', y' and z' are neglected. It appears moreover from the solution that there is only one way in which these conditions can be satisfied for a given system of co-ordinate axes. Let us take new rectangular axes Ox, Oy, Oz, and let U, V, W be the resolved parts along these axes of the velocities U, V, W, and x′, y′, z′, the relative co-ordinates of P'; then

$$U = \mathrm{U} \cdot \cos(x\mathrm{x}) + \mathrm{V} \cdot \cos(x\mathrm{y}) + \mathrm{W} \cdot \cos(x\mathrm{z}),$$
$$\mathrm{d}x' = \cos(x\mathrm{x})\,\mathrm{dx}' + \cos(x\mathrm{y})\,\mathrm{dy}' + \cos(x\mathrm{z})\,\mathrm{dz}',\ \&c.;$$

whence, taking account of the well-known relations between the cosines involved in these equations, we easily find

$$U\,\mathrm{d}x' + V\,\mathrm{d}y' + W\,\mathrm{d}z' = \mathrm{U}\,\mathrm{dx}' + \mathrm{V}\,\mathrm{dy}' + \mathrm{W}\,\mathrm{dz}'.$$

It appears therefore that the relative velocities U, V, W, which remain after eliminating a certain angular velocity, are such that $\mathrm{U}\,\mathrm{dx}' + \mathrm{V}\,\mathrm{dy}' + \mathrm{W}\,\mathrm{dz}'$ is ultimately an exact differential. Hence the values of U, V, W are the same as would have been obtained from Equation 2.2 applied directly to the new axes, whence the truth of the proposition enunciated at the head of this chapter is manifest.
The motion corresponding to the velocities $U_\prime$, $V_\prime$, $W_\prime$ may be further decomposed into a motion of dilatation, positive or negative, which is alike in all directions, and two motions which I shall call *motions of shifting*, each of the latter being in two dimensions, and not affecting the density. For let δ be the rate of linear extension corresponding to a uniform dilatation; let $\sigma x'_\prime$, $-\sigma y'_\prime$ be the velocities parallel to $x_\prime$, $y_\prime$, corresponding to a motion of shifting parallel to the plane $x_\prime y_\prime$, and let $\sigma' x'_\prime$, $-\sigma' z'_\prime$ be the velocities parallel to $x_\prime$, $z_\prime$, corresponding to a similar motion of shifting parallel to the plane $x_\prime z_\prime$. The velocities parallel to $x_\prime$, $y_\prime$, $z_\prime$ respectively corresponding to the quantities δ, σ and σ' will be $(\delta + \sigma + \sigma') \cdot x'_\prime$, $(\delta - \sigma) \cdot y'_\prime$, $(\delta - \sigma') \cdot z'_\prime$, and equating these to $U_\prime$, $V_\prime$, $W_\prime$ we shall get

$$\delta = \frac{1}{3} \cdot (e' + e'' + e''') \ ,$$
$$\sigma = \frac{1}{3} \cdot (e' + e''' - 2e'') \ ,$$
$$\sigma' = \frac{1}{3} \cdot (e' + e'' - 2e''') \ .$$

Hence the most general instantaneous motion of an elementary portion of a fluid is compounded of a motion of translation, a motion of rotation, a motion of uniform dilatation, and two motions of shifting of the kind just mentioned.

3. Having determined the nature of the most general instantaneous motion of an element of a fluid, we are now prepared to consider the normal pressures and tangential forces called into play by the relative displacements of the particles. Let p be the pressure which would exist about the point P if the neighbouring molecules were in a state of relative equilibrium: let $p + p_\prime$ be the normal pressure, and $t_\prime$ the tangential action, both referred to a unit of surface, on a plane passing through P and having a given direction. By the hypotheses of Art. 1., page 73 the quantities $p_\prime$, $t_\prime$ will be independent of the angular velocities ω', ω'', ω''', depending only on the residual relative velocities U, V, W, or, which comes to the same, on e', e'' and e''', or on σ, σ' and δ. Since this residual motion is symmetrical with respect to the axes of extension, it follows that if the plane considered is perpendicular to any one of these axes the tangential action on it is zero, since there is no more reason why it should act in one direction rather than in the opposite; for by the hypotheses of Art. 1., page 73 the change of density and temperature about the point P is to be neglected, the constitution of the fluid being ultimately uniform about that point. Denoting then by $p + p'$, $p + p''$, $p + p'''$ the pressures on planes

perpendicular to the axes of $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, we must have

$$p' = \phi\,(e', e'', e''')\ ,$$
$$p'' = \phi\,(e'', e''', e')\ ,$$
$$p''' = \phi\,(e''', e', e'')\ ,$$

$\phi\,(e', e'', e''')$ denoting a function of e', e'' and e''' which is symmetrical with respect to the two latter quantities. The question is now to determine, on whatever may seem the most probable hypothesis, the form of the function ϕ.

Let us first take the simpler case in which there is no dilatation, and only one motion of shifting, or in which $e'' = -e'$, $e''' = 0$, and let us consider what would take place if the fluid consisted of smooth molecules acting on each other by actual contact. On this supposition, it is clear, considering the magnitude of the pressures acting on the molecules compared with their masses, that they would be sensibly in a position of relative equilibrium, except when the equilibrium of any one of them became impossible from the displacement of the adjoining ones, in which case the molecule in question would start into a new position of equilibrium. This start would cause a corresponding displacement in the molecules immediately about the one which started, and this disturbance would be propagated immediately in all directions, the nature of the displacement however being different in different directions, and would soon become insensible. During the continuance of this disturbance, the pressure on a small plane drawn through the element considered would not be the same in all directions, nor normal to the plane: or, which comes to the same, we may suppose a uniform normal pressure p to act, together with a normal pressure $p_{\prime\prime}$ and a tangential force $t_{\prime\prime}$, $p_{\prime\prime}$ and $t_{\prime\prime}$ being forces of great intensity and short duration, that is being of the nature of impulsive forces. As the number of molecules comprised in the element considered has been supposed extremely great, we may take a time τ so short that all summations with respect to such intervals of time may be replaced without sensible error by integrations, and yet so long that a very great number of starts shall take place in it. Consequently we have only to consider the average effect of such starts, and moreover we may without sensible error replace the impulsive forces such as $p_{\prime\prime}$ and $t_{\prime\prime}$, which succeed one another with great rapidity, by continuous forces. For planes perpendicular to the axes of extension these continuous forces will be the normal pressures p', p'', p'''.

Let us now consider a motion of shifting differing from the former only in having e' increased in the ratio of m to 1. Then, if we suppose each start completed before the starts which would be sensibly affected by it are begun, it is clear that the same series of starts will take place in the second case as in the first, but at intervals of time which are less in the ratio of 1 to m. Consequently the continuous pressures by which the impulsive actions due to these starts may be replaced must be increased in the ratio of m to 1. Hence the pressures p', p'', p''' must be proportional to e',

or we must have

$$p' = C \cdot e' ,$$
$$p'' = C' \cdot e' ,$$
$$p''' = C'' \cdot e' .$$

It is natural to suppose that these formulæ hold good for negative as well as positive values of e'. Assuming this to be true, let the sign of e' be changed. This comes to interchanging x and y, and consequently p''' must remain the same, and p' and p'' must be interchanged. We must therefore have $C'' = 0$, $C' = -C$. Putting then $C = -2\mu$ we have

$$p' = -2\mu \cdot e' ,$$
$$p'' = 2\mu \cdot e' ,$$
$$p''' = 0 .$$

It has hitherto been supposed that the molecules of a fluid are in actual contact. We have every reason to suppose that this is not the case. But precisely the same reasoning will apply if they are separated by intervals as great as we please compared with their magnitudes, provided only we suppose the force of restitution called into play by a small displacement of *any one* molecule to be very great.
Let us now take the case of two motions of shifting which coexist, and let us suppose $e' = \sigma + \sigma'$, $e'' = -\sigma$, $e''' = -\sigma'$. Let the small time τ be divided into $2n$ equal portions, and let us suppose that in the first interval a shifting motion corresponding to $e' = 2\sigma$, $e'' = -2\sigma$ takes place parallel to the plane $x_{\prime}y_{\prime}$, and that in the second interval a shifting motion corresponding to $e' = 2\sigma'$, $e''' = -2\sigma'$ takes place parallel to the plane at $x_{\prime}z_{\prime}$, and so on alternately. On this supposition it is clear that if we suppose the time $\frac{\tau}{2n}$ to be extremely small, the continuous forces by which the effect of the starts may be replaced will be

$$p' = -2\mu \cdot (\sigma + \sigma') ,$$
$$p'' = 2\mu \cdot \sigma ,$$
$$p''' = 2\mu \cdot \sigma' .$$

By supposing n indefinitely increased, we might make the motion considered approach as near as we please to that in which the two motions of shifting coexist; but we are not at liberty to do so, for in order to apply the above reasoning we must suppose the time $\frac{\tau}{2n}$ to be so large that the average effect of the starts which occur in it may be taken. Consequently it must be taken as an additional assumption, and not a matter of absolute demonstration, that the effects of the two motions of shifting are superimposed.

Hence if $\delta = 0$, i.e. if $e' + e'' + e''' = 0$, we shall have in general

$$\left.\begin{aligned} p' &= -2\mu \cdot e' , \\ p'' &= -2\mu \cdot e'' , \\ p''' &= -2\mu \cdot e''' . \end{aligned}\right\} \tag{2.5}$$

It was by this hypothesis of starts that I first arrived at these equations, and the differential equations of motion which result from them. On reading Poisson's memoir however, to which I shall have occasion to refer in chapter 5, I was led to reflect that however intense we may suppose the molecular forces to be, and however near we may suppose the molecules to be to their positions of relative equilibrium, we are not therefore at liberty to suppose them *in* those positions, and consequently not at liberty to suppose the pressure equal in all directions in the intervals of time between the starts. In fact, by supposing the molecular forces indefinitely increased, retaining the same ratios to each other, we may suppose the displacements of the molecules from their positions of relative equilibrium indefinitely diminished, but on the other hand the force of restitution called into action by a given displacement is indefinitely increased in the same proportion. But be these displacements what they may, we know that the forces of restitution make equilibrium with forces equal and opposite to the effective forces; and in calculating the effective forces we may neglect the above displacements, or suppose the molecules to move in the paths in which they would move if the shifting motion took place with indefinite slowness. Let us first consider a single motion of shifting, or one for which $e'' = -e'$, $e''' = 0$, and let $p_{,}$ and $t_{,}$ denote the same quantities as before. If we now suppose e' increased in the ratio of m to 1, all the effective forces will be increased in that ratio, and consequently $p_{,}$ and $t_{,}$ will be increased in the same ratio. We may deduce the values of p', p'' and p''' just as before, and then pass by the same reasoning to the case of two motions of shifting which coexist, only that in this case the reasoning will be demonstrative, since we *may* suppose the time $\frac{\tau}{2n}$ indefinitely diminished. If we suppose the state of things considered in this chapter to exist along with the motions of starting already considered, it is easy to see that the expressions for p', p'' and p''' will still retain the same form.
There remains yet to be considered the effect of the dilatation. Let us first suppose the dilatation to exist without any shifting: then it is easily seen that the relative motion of the fluid at the point considered is the same in all directions. Consequently the only effect which such a dilatation could have would be to introduce a normal pressure $p_{,}$, alike in all directions, in addition to that due to the action of the molecules supposed to be in a state of relative equilibrium. Now the pressure $p_{,}$ could only arise from the aggregate of the molecular actions called into play by the displacements of the molecules from their positions of relative equilibrium; but since these displacements take place, on an average, indifferently in all directions, it follows that the actions of which $p_{,}$ is composed neutralize each other, so that $p_{,} = 0$.

The same conclusion might be drawn from the hypothesis of starts, supposing, as it is natural to do, that each start calls into action as much increase of pressure in some directions as diminution of pressure in others.
If the motion of uniform dilatation coexists with two motions of shifting, I shall suppose, for the same reason as before, that the effects of these different motions are superimposed. Hence subtracting δ from each of the three quantities e', e'' and e''', and putting the remainders in the place of e', e'' and e''' in Equation 2.5, we have

$$\left.\begin{aligned} p' &= \frac{2}{3}\mu \cdot (e'' + e''' - 2e') \ , \\ p'' &= \frac{2}{3}\mu \cdot (e''' + e' - 2e'') \ , \\ p''' &= \frac{2}{3}\mu \cdot (e' + e'' - 2e''') \ . \end{aligned}\right\} \tag{2.6}$$

If we had started with assuming $\phi(e', e'', e''')$ to be a linear function of e', e'' and e''', avoiding all speculation as to the molecular constitution of a fluid, we should have had at once

$$p' = Ce' + C' \cdot (e'' + e''') \, ,$$

since p' is symmetrical with respect to e'' and e'''; or, changing the constants,

$$p' = \frac{2}{3}\mu \cdot (e'' + e''' - 2e') + \kappa \cdot (e' + e'' + e''') \ .$$

The expressions for p'' and p''' would be obtained by interchanging the requisite quantities. Of course we may at once put $\kappa = 0$ if we assume that in the case of a uniform motion of dilatation the pressure at any instant depends only on the actual density and temperature at that instant, and not on the rate at which the former changes with the time. In most cases to which it would be interesting to apply the theory of the friction of fluids the density of the fluid is either constant, or may without sensible error be regarded as constant, or else changes slowly with the time. In the first two cases the results would be the same, and in the third case nearly the same, whether κ were equal to zero or not. Consequently, if theory and experiment should in such cases agree, the experiments must not be regarded as confirming that part of the theory which relates to supposing κ to be equal to zero.

4. It will be easy now to determine the oblique pressure, or resultant of the normal pressure and tangential action, on any plane. Let us first consider a plane drawn through the point P parallel to the plane yz. Let $Ox_\prime$ make with the axes of x, y, z angles whose cosines are l', m', n'; let l'', m'', n'' be the same for $Oy_\prime$, and l''', m''', n''' the same for $Oz_\prime$. Let P_1 be the pressure, and (xty), (xtz) the resolved parts, parallel to y, z respectively, of the tangential force on the plane considered, all referred to a unit of surface, (xty) being reckoned positive when the part of the

fluid towards $-x$ urges that towards $+x$ in the positive direction of y, and similarly for (xtz). Consider the portion of the fluid comprised within a tetrahedron having its vertex in the point P, its base parallel to the plane yz, and its three sides parallel to the planes $x_\prime y_\prime$, $y_\prime z_\prime$, $z_\prime x_\prime$ respectively. Let A be the area of the base, and therefore $l'A$, $l''A$, $l'''A$ the areas of the faces perpendicular to the axes of $x_\prime$, $y_\prime$, $z_\prime$. By D'Alembert's principle, the pressures and tangential actions on the faces of this tetrahedron, the moving forces arising from the external attractions, not including the molecular forces, and forces equal and opposite to the effective moving forces will be in equilibrium, and therefore the sums of the resolved parts of these forces in the directions of x, y and z will each be zero. Suppose now the dimensions of the tetrahedron indefinitely diminished, then the resolved parts of the external, and of the effective moving forces will vary ultimately as the cubes, and those of the pressures and tangential forces on the sides as the squares of homologous lines. Dividing therefore the three equations arising from equating to zero the resolved parts of the above forces by A, and taking the limit, we have

$$\begin{aligned} P_1 &= \sum l'^2 \cdot (p + p') \ , \\ (xty) &= \sum l'm' \cdot (p + p') \ , \\ (xtz) &= \sum l'n' \cdot (p + p') \ , \end{aligned}$$

the sign $\sum$ denoting the sum obtained by taking the quantities corresponding to the three axes of extension in succession. Putting for p', p'' and p''' their values given by Equation 2.6, putting $e' + e'' + e''' = 3\delta$, and observing that $\sum l'^2 = 1$, $\sum l'm' = 0$, $\sum l'n' = 0$, the above equations become

$$\begin{aligned} P_1 &= p - 2\mu \cdot \sum l'^2 e' + 2\mu\delta \, , \\ (xty) &= -2\mu \cdot \sum l'm'e' \, , \\ (xtz) &= -2\mu \cdot \sum l'n'e' \, . \end{aligned}$$

The method of determining the pressure on any plane from the pressures on three planes at right angles to each other, which has just been given, has already been employed by MM. Cauchy and Poisson.

The most direct way of obtaining the values of $\sum l'^2 e'$ &c. would be to express l', m' and n' in terms of e' by any two of Equation 2.3, in which x', y', z' are proportional to l', m', n', together with the equation $l'^2 + m'^2 + n'^2 = 1$, and then to express the resulting symmetrical function of the roots of the cubic Equation 2.4 in terms of the coefficients. But this method would be excessively laborious, and need not be resorted to. For after eliminating the angular motion of the element of fluid considered the remaining velocities are $e'x'_\prime$, $e''y'_\prime$, $e'''z'_\prime$, parallel to the axes of $x_\prime$, $y_\prime$, $z_\prime$. The sum of the resolved parts of these parallel to the axis of x is

$l'e'x'_{,} + l''e''y'_{,} + l'''e'''z'_{,}$. Putting for $x'_{,}$, $y'_{,}$, $z'_{,}$ their values $l'x' + m'y' + n'z'$ &c., the above sum becomes

$$x' \sum l'^2 e' + y' \sum l'm'e' + z' \sum l'n'e' ;$$

but this sum is the same thing as the velocity U in Equation 2.2, and therefore we have

$$\begin{aligned} \sum l'^2 e' &= \frac{\mathrm{d}u}{\mathrm{d}x} , \\ \sum l'm'e' &= \frac{1}{2} \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}x} \right) , \\ \sum l'n'e' &= \frac{1}{2} \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}x} \right) . \end{aligned}$$

It may also be very easily proved directly that the value of 3δ, the rate of cubical dilatation, satisfies the equation

$$3\delta = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z} . \tag{2.7}$$

Let P_2, (ytz), (ytx) be the quantities referring to the axis of y, and P_3, (ztx), (zty) those referring to the axis of z, which correspond to P_1 (xty), (xtz), referring to the axis of x. Then we see that $(ytz) = (zty)$, $(ztx) = (xtz)$, $(xty) = (ytx)$. Denoting these three quantities by T_1, T_2, T_3, and making the requisite substitutions and interchanges, we have

$$\left.\begin{aligned} P_1 &= p - 2\mu \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}x} - \delta \right) , \\ P_2 &= p - 2\mu \cdot \left(\frac{\mathrm{d}v}{\mathrm{d}y} - \delta \right) , \\ P_3 &= p - 2\mu \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}z} - \delta \right) , \\ T_1 &= -\mu \cdot \left(\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y} \right) , \\ T_2 &= -\mu \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z} \right) , \\ T_3 &= -\mu \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}x} \right) . \end{aligned}\right\} \tag{2.8}$$

It may also be useful to know the components, parallel to x, y, z, of the oblique pressure on a plane passing through the point P, and having a given direction. Let l, m, n be the cosines of the angles which a normal to the given plane makes with

the axes of x, y, z; let P, Q, R be the components, referred to a unit of surface, of the oblique pressure on this plane, P, Q, R being reckoned positive when the part of the fluid in which is situated the normal to which l, m and n refer is urged by the other part in the positive directions of x, y, z, when l, m and n are positive. Then considering as before a tetrahedron of which the base is parallel to the given plane, the vertex in the point P, and the sides parallel to the co-ordinate planes, we shall have

$$\left.\begin{aligned} P &= l \cdot P_1 + m \cdot T_3 + n \cdot T_2 \,, \\ Q &= l \cdot T_3 + m \cdot P_2 + n \cdot T_1 \,, \\ R &= l \cdot T_2 + m \cdot T_1 + n \cdot P_3 \,. \end{aligned}\right\} \tag{2.9}$$

In the simple case of a sliding motion for which $u = 0$, $v = f(x)$, $w = 0$, the only forces, besides the pressure p, which act on planes parallel to the co-ordinate planes are the two tangential forces T_3, the value of which in this case is $-\mu \cdot \frac{\mathrm{d}v}{\mathrm{d}x}$; In this case it is easy to shew that the axes of extension are, one of them parallel to Oz, and the two others in a plane parallel to xy, and inclined at angles of 45° to Ox. We see also that it is necessary to suppose μ to be positive, since otherwise the tendency of the forces would be to increase the relative motion of the parts of the fluid, and the equilibrium of the fluid would be unstable.

5. Having found the pressures about the point P on planes parallel to the co-ordinate planes, it will be easy to form the equations of motion. Let X, Y, Z be the resolved parts, parallel to the axes, of the external force, not including the molecular force; let ρ be the density, t the time. Consider an elementary parallelepiped of the fluid, formed by planes parallel to the co-ordinate planes, and drawn through the point (x, y, z) and the point $(x + \Delta x, y + \Delta y, z + \Delta z)$. The mass of this element will be ultimately $\rho \cdot \Delta x \Delta y \Delta z$, and the moving force parallel to x arising from the external forces will be ultimately $\rho X \cdot \Delta x \Delta y \Delta z$; the effective moving force parallel to x will be ultimately $\rho \cdot \frac{\mathrm{D}u}{\mathrm{D}t} \cdot \Delta x \Delta y \Delta z$, where D is used, as it will be in the rest of this paper, to denote differentiation in which the independent variables are t and three parameters of the particle considered, (such for instance as its initial co-ordinates,) and not t, x, y, z.[1] It is easy also to shew that the moving force acting on the element considered arising from the oblique pressures

[1] In continuum mechanics, or like here in the theory of the motion of elastic solids, the material derivative $\frac{\mathrm{D}}{\mathrm{D}t}$ describes the time rate of change of some physical quantity (like heat or momentum) *of a material element* or with STOKES, *of the particle considered* that is subjected to a space-and-time-dependent macroscopic velocity field (motion). For example, in fluid dynamics, the velocity field is the flow velocity, and the quantity of interest might be the temperature of the fluid. In which case, the material derivative then describes the temperature change of a certain fluid parcel with time, as it flows along its pathline (trajectory) with its cartesian co-ordinates x, y, z.
The material derivative is defined for any tensor field y that is macroscopic, with the sense that

on the faces is ultimately

$$\left(\frac{\mathrm{d}P}{\mathrm{d}x}+\frac{\mathrm{d}T_3}{\mathrm{d}y}+\frac{\mathrm{d}T_2}{\mathrm{d}z}\right)\cdot\Delta x\Delta y\Delta z\,,$$

acting in the negative direction. Hence we have by D'Alembert's principle

$$\left.\begin{aligned}\rho\cdot\left(\frac{\mathrm{D}u}{\mathrm{D}t}-X\right)+\frac{\mathrm{d}P_1}{\mathrm{d}x}+\frac{\mathrm{d}T_3}{\mathrm{d}y}+\frac{\mathrm{d}T_2}{\mathrm{d}z}=0\,,\\ \rho\cdot\left(\frac{\mathrm{D}v}{\mathrm{D}t}-Y\right)+\frac{\mathrm{d}T_3}{\mathrm{d}x}+\frac{\mathrm{d}P_2}{\mathrm{d}y}+\frac{\mathrm{d}T_1}{\mathrm{d}z}=0\,,\\ \rho\cdot\left(\frac{\mathrm{D}w}{\mathrm{D}t}-Z\right)+\frac{\mathrm{d}T_2}{\mathrm{d}x}+\frac{\mathrm{d}T_1}{\mathrm{d}y}+\frac{\mathrm{d}P_3}{\mathrm{d}z}=0\,,\end{aligned}\right\}\tag{2.10}$$

in which equations we must put for $\frac{\mathrm{D}u}{\mathrm{D}t}$ its value

$$=\frac{\mathrm{d}u}{\mathrm{d}t}+u\frac{\mathrm{d}u}{\mathrm{d}x}+v\frac{\mathrm{d}u}{\mathrm{d}y}+w\frac{\mathrm{d}u}{\mathrm{d}z}\,,$$

and similarly for $\frac{\mathrm{D}v}{\mathrm{D}t}$ and $\frac{\mathrm{D}w}{\mathrm{D}t}$. Equation 2.10, the ordinary equation of continuity, as it is called,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\frac{\mathrm{d}\rho u}{\mathrm{d}x}+\frac{\mathrm{d}\rho v}{\mathrm{d}y}+\frac{\mathrm{d}\rho w}{\mathrm{d}z}=0\,,\tag{2.11}$$

which expresses the condition that there is no generation or destruction of mass in the interior of a fluid, the equation connecting p and ρ, or in the case of an incompressible fluid the equivalent equation $\frac{\mathrm{D}\rho}{\mathrm{D}t}=0$, and the equation for the propagation of heat, if we choose to take account of that propagation, are the only equations to be satisfied at every point of the interior of the fluid mass.
As it is quite useless to consider cases of the utmost degree of generality, I shall

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it depends only on position and time co-ordinates, $y=y(\mathbf{x},t)$:

$$\frac{\mathrm{D}y}{\mathrm{D}t}\equiv\frac{\partial y}{\partial t}+\mathbf{u}\cdot\nabla y\,,$$

where ∇y is the covariant derivative of the tensor, and $\mathbf{u}(\mathbf{x},t)$ is the flow velocity. For a macroscopic scalar field $\phi(\mathbf{x},t)$ the definition becomes

$$\frac{\mathrm{D}\phi}{\mathrm{D}t}\equiv\frac{\partial\phi}{\partial t}+\mathbf{u}\cdot\nabla\phi\,,$$

and therefore we get for the convective term in case of a scalar field in a three-dimensional Cartesian coordinate system $\mathbf{x}=(x,y,z)$, the components of the velocity $\mathbf{u}$ being u, v, w,:

$$\mathbf{u}\cdot\nabla\phi=u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}+w\frac{\partial\phi}{\partial z}\,.$$

suppose the fluid to be homogeneous, and of a uniform temperature throughout, except in so far as the temperature may be raised by sudden compression in the case of small vibrations. Hence in Equation 2.10 μ may be supposed to be constant as far as regards the temperature; for, in the case of small vibrations, the terms introduced by supposing it to vary with the temperature would involve the square of the velocity, which is supposed to be neglected. If we suppose μ to be independent of the pressure also, and substitute in Equation 2.10 the values of P_1, &c. given by Equation 2.8, the former equations become

$$\left.\begin{aligned}
&\rho \cdot \left(\frac{\mathrm{D}u}{\mathrm{D}t} - X\right) + \frac{\mathrm{d}p}{\mathrm{d}x} - \mu \cdot \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2u}{\mathrm{d}y^2} + \frac{\mathrm{d}^2u}{\mathrm{d}z^2}\right) \\
&- \frac{\mu}{3} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z}\right) = 0\,, \\
&\rho \cdot \left(\frac{\mathrm{D}v}{\mathrm{D}t} - Y\right) + \frac{\mathrm{d}p}{\mathrm{d}y} - \mu \cdot \left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \frac{\mathrm{d}^2v}{\mathrm{d}y^2} + \frac{\mathrm{d}^2v}{\mathrm{d}z^2}\right) \\
&- \frac{\mu}{3} \cdot \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z}\right) = 0\,, \\
&\rho \cdot \left(\frac{\mathrm{D}w}{\mathrm{D}t} - Z\right) + \frac{\mathrm{d}p}{\mathrm{d}z} - \mu \cdot \left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + \frac{\mathrm{d}^2w}{\mathrm{d}y^2} + \frac{\mathrm{d}^2w}{\mathrm{d}z^2}\right) \\
&- \frac{\mu}{3} \cdot \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z}\right) = 0\,,
\end{aligned}\right\} \tag{2.12}$$

Let us now consider in what cases it is allowable to suppose μ to be independent of the pressure. It has been concluded by Dubuat, from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure. The total retardation depends, partly on the friction of the water against the sides of the pipe or canal, and partly on the mutual friction, or tangential action, of the different portions of the water. Now if these two parts of the whole retardation were separately variable with p, it is very unlikely that they should when combined give a result independent of p. The amount of the internal friction of the water depends on the value of μ. I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of the pressure. On this supposition, we have from Equation 2.11

and Equation 2.12

$$\left.\begin{aligned}
\rho\cdot\left(\frac{\mathrm{D}u}{\mathrm{D}t}-X\right)+\frac{\mathrm{d}p}{\mathrm{d}x}-\mu\cdot\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+\frac{\mathrm{d}^2u}{\mathrm{d}y^2}+\frac{\mathrm{d}^2u}{\mathrm{d}z^2}\right)=0\,,\\
\rho\cdot\left(\frac{\mathrm{D}v}{\mathrm{D}t}-Y\right)+\frac{\mathrm{d}p}{\mathrm{d}y}-\mu\cdot\left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+\frac{\mathrm{d}^2v}{\mathrm{d}y^2}+\frac{\mathrm{d}^2v}{\mathrm{d}z^2}\right)=0\,,\\
\rho\cdot\left(\frac{\mathrm{D}w}{\mathrm{D}t}-Z\right)+\frac{\mathrm{d}p}{\mathrm{d}z}-\mu\cdot\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+\frac{\mathrm{d}^2w}{\mathrm{d}y^2}+\frac{\mathrm{d}^2w}{\mathrm{d}z^2}\right)=0\,,
\end{aligned}\right\}\qquad(2.13)$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\mathrm{d}v}{\mathrm{d}y}+\frac{\mathrm{d}w}{\mathrm{d}z}=0\,.$$

These equations are applicable to the determination of the motion of water in pipes and canals, to the calculation of the effect of friction on the motions of tides and waves, and such questions.

If the motion is very small, so that we may neglect the square of the velocity, we may put $\frac{\mathrm{D}u}{\mathrm{D}t}=\frac{\mathrm{d}u}{\mathrm{d}t}$, &c. in Equation 2.13. The equations thus simplified are applicable to the determination of the motion of a pendulum oscillating in water, or of that of a vessel filled with water and made to oscillate. They are also applicable to the determination of the motion of a pendulum oscillating in air, for in this case we may, with hardly any error, neglect the compressibility of the air.

The case of the small vibrations by which sound is propagated in a fluid, whether a liquid or a gas, is another in which $\frac{\mathrm{d}\mu}{\mathrm{d}p}$ may be neglected. For in the case of a liquid reasons have been shewn for supposing μ to be independent of p, and in the case of a gas we may neglect $\frac{\mathrm{d}\mu}{\mathrm{d}p}$, if we neglect the small change in the value of μ, arising from the small variation of pressure due to the forces X, Y, Z.

6. Besides the equations which must hold good at any point in the interior of the mass, it will be necessary to form also the equations which must be satisfied at its boundaries. Let M be a point in the boundary of the fluid. Let a normal to the surface at M, drawn on the outside of the fluid, make with the axes angles whose cosines are l, m, n. Let P', Q', R' be the components of the pressure of the fluid about M on the solid or fluid with which it is in contact, these quantities being reckoned positive when the fluid considered presses the solid or fluid outside it in the positive directions of x, y, z, supposing l, m and n positive. Let S be a very small element of the surface about M, which will be ultimately plane, S' a plane parallel and equal to S, and directly opposite to it, taken within the fluid. Let the distance between S and S' be supposed to vanish in the limit compared with the breadth of S, a supposition which may be made if we neglect the effect of the curvature of the surface at M; and let us consider the forces acting on the element of fluid comprised between S and S', and the motion of this element. If we suppose Equation 2.8 to hold good to within an insensible distance from the surface of the fluid, we shall evidently have forces ultimately equal to PS, QS,

RS, (P, Q and R being given by Equation 2.9,) acting on the inner side of the element in the positive directions of the axes, and forces ultimately equal to $P'S$, $Q'S$, $R'S$ acting on the outer side in the negative directions. The moving forces arising from the external forces acting on the element, and the effective moving forces will vanish in the limit compared with the forces PS, &c.: the same will be true of the pressures acting about the edge of the element, if we neglect capillary attraction, and all forces of the same nature. Hence, taking the limit, we shall have

$$P' = P\,, \quad Q' = Q\,, \quad R' = R\,.$$

The method of proceeding will be different according as the bounding surface considered is a free surface, the surface of a solid, or the surface of separation of two fluids, and it will be necessary to consider these cases separately. Of course the surface of a liquid exposed to the air is really the surface of separation of two fluids, but it may in many cases be regarded as a free surface if we neglect the inertia of the air: it may always be so regarded if we neglect the friction of the air as well as its inertia.

Let us first take the case of a free surface exposed to a pressure Π, which is supposed to be the same at all points, but may vary with the time; and let $L = 0$ be the equation to the surface. In this case we shall have $P' = l \cdot \Pi$, $Q' = m \cdot \Pi$, $R' = n \cdot \Pi$; and putting for P, Q, R their values given by Equation 2.9, and for P_1 &c. their values given by Equation 2.8, and observing that in this case $\delta = 0$, we shall have

$$\left.\begin{aligned}
l \cdot (\Pi - p) + \mu \cdot \left\{ 2l \cdot \frac{\mathrm{d}u}{\mathrm{d}x} + m \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}x} \right) + n \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}x} \right) \right\} = 0 \\
m \cdot (\Pi - p) + \mu \cdot \left\{ l \cdot \left(\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y} \right) + 2m \cdot \frac{\mathrm{d}v}{\mathrm{d}y} + n \cdot \left(\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}y} \right) \right\} = 0 \\
n \cdot (\Pi - p) + \mu \cdot \left\{ l \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z} \right) + m \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}z} \right) + 2n \cdot \frac{\mathrm{d}w}{\mathrm{d}z} \right\} = 0
\end{aligned}\right\} \quad (2.14)$$

in which equations l, m, n will have to be replaced by $\frac{\mathrm{d}L}{\mathrm{d}x}$, $\frac{\mathrm{d}L}{\mathrm{d}y}$, $\frac{\mathrm{d}L}{\mathrm{d}z}$, to which they are proportional.

If we choose to take account of capillary attraction, we have only to diminish the pressure Π by the quantity $H\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$, where H is a positive constant depending on the nature of the fluid, and r_1, r_2 are the principal radii of curvature at the point considered, reckoned positive when the fluid is concave outwards. Equation 2.14 with the ordinary equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} + u\frac{\mathrm{d}L}{\mathrm{d}x} + v\frac{\mathrm{d}L}{\mathrm{d}y} + w\frac{\mathrm{d}L}{\mathrm{d}z} = 0\,, \quad (2.15)$$

are the conditions to be satisfied for points at the free surface. Equation 2.14 are for such points what the three equations of motion are for internal points, and

Equation 2.15 is for the former what Equation 2.11 is for the latter, expressing in fact that there is no generation or destruction of fluid at the free surface.
Equation 2.14 admit of being differently expressed, in a way which may sometimes be useful. If we suppose the origin to be in the point considered, and the axis of z to be the external normal to the surface, we have $l = m = 0$, $n = l$, and the equations become

$$\left.\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z} &= 0\,, \\ \frac{\mathrm{d}w}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}z} &= 0\,, \\ \Pi - p + 2\mu \cdot \frac{\mathrm{d}w}{\mathrm{d}z} &= 0\,. \end{aligned}\right\} \tag{2.16}$$

The relative velocity parallel to z of a point $(x', y', 0)$ in the free surface, indefinitely near the origin, is $\frac{\mathrm{d}w}{\mathrm{d}x}x' + \frac{\mathrm{d}w}{\mathrm{d}y}y'$: hence we see that $\frac{\mathrm{d}w}{\mathrm{d}x}$, $\frac{\mathrm{d}w}{\mathrm{d}y}$ are the angular velocities, reckoned from x to z and from y to z respectively, of an element of the free surface. Subtracting the linear velocities due to these angular velocities from the relative velocities of the point (x', y', z'), and calling the remaining relative velocities U, V, W, we shall have

$$\begin{aligned} U &= \frac{\mathrm{d}u}{\mathrm{d}x}x' + \frac{\mathrm{d}u}{\mathrm{d}y}y' + \left(\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}x}\right)z'\,, \\ V &= \frac{\mathrm{d}v}{\mathrm{d}x}x' + \frac{\mathrm{d}v}{\mathrm{d}y}y' + \left(\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y}\right)z'\,, \\ W &= \frac{\mathrm{d}w}{\mathrm{d}z}z'\,. \end{aligned}$$

Hence we see that the first two of Equation 2.16 express the conditions that $\frac{\mathrm{d}U}{\mathrm{d}z'} = 0$ and $\frac{\mathrm{d}V}{\mathrm{d}z'} = 0$, which are evidently the conditions to be satisfied in order that there may be no sliding motion in a direction parallel to the free surface. It would be easy to prove that these are the conditions to be satisfied in order that the axis of z may be an axis of extension.
The next case to consider is that of a fluid in contact with a solid. The condition which first occurred to me to assume for this case was, that the film of fluid immediately in contact with the solid did not move relatively to the surface of the solid.[2] I was led to try this condition from the following considerations. According to the hypotheses adopted, if there was a very large relative motion of the fluid particles immediately about any imaginary surface dividing the fluid, the tangential forces called into action would be very large, so that the amount of relative motion would be rapidly diminished. Passing to the limit, we might suppose that if at

[2]An indirect supposition of the existence of a boundary layer (`https://en.wikipedia.org/wiki/Boundary_layer`).

any instant the velocities altered discontinuously in passing across any imaginary surface, the tangential force called into action would immediately destroy the finite relative motion of particles indefinitely close to each other, so as to render the motion continuous; and from analogy the same might be supposed to be true for the surface of junction of a fluid and solid. But having calculated, according to the conditions which I have mentioned, the discharge of long straight circular pipes and rectangular canals, and compared the resulting formulæ with some of the experiments of Bossut and Dubuat, I found that the formulæ did not at all agree with experiment. I then tried Poisson's conditions in the case of a circular pipe, but with no better success. In fact, it appears from experiment that the tangential force varies nearly as the square of the velocity with which the fluid flows past the surface of a solid, at least when the velocity is not very small. It appears however from experiments on pendulums that the total friction varies as the first power of the velocity, and consequently we may suppose that Poisson's conditions, which include as a particular case those which I first tried, hold good for very small velocities. I proceed therefore to deduce these conditions in a manner conformable with the views explained in this paper.

First, suppose the solid at rest, and let $L = 0$ be the equation to its surface. Let M' be a point within the fluid, at an insensible distance h from M. Let ϖ be the pressure which would exist about M if there were no motion of the particles in its neighbourhood, and let $p_{\prime}$ be the additional normal pressure, and $t_{\prime}$ the tangential force, due to the relative velocities of the particles, both with respect to one another and with respect to the surface of the solid. If the motion is so slow that the starts take place independently of each other, on the hypothesis of starts, or that the molecules are very nearly in their positions of relative equilibrium, and if we suppose as before that the effects of different relative velocities are superimposed, it is easy to shew that $p_{\prime}$ and $t_{\prime}$ are linear functions of u, v, w and their differential coefficients with respect to x, y, and z; u, v, &c. denoting here the velocities of the fluid about the point M', in the expressions for which however the co-ordinates of M may be used for those of M', since h is neglected. Now the relative velocities about the points M and M' depending on $\frac{\mathrm{d}u}{\mathrm{d}x}$ &c. are comparable with $\frac{\mathrm{d}u}{\mathrm{d}x} \cdot h$, while those depending on u, v and w are comparable with these quantities, and therefore in considering the action of the fluid on the solid it is only necessary to consider the quantities u, v and w. Now since, neglecting h, the velocity at M' is tangential to the surface at M, u, v, and w are the components of a certain velocity V tangential to the surface. The pressure $p_{\prime}$ must be zero; for changing the signs u, v, and w the circumstances concerned in its production remain the same, whereas its analytical expression changes sign. The tangential force at M will be in the direction of V, and proportional to it, and consequently its components along the axes of x, y, z will be proportional to u, v, w. Reckoning the tangential force positive when, l, m, and n being positive, the solid is urged in the positive directions of x, y, z, the resolved parts of the tangential force will therefore be νu, νv, νw, where ν must

evidently be positive, since the effect of the forces must be to check the relative motion of the fluid and solid. The normal pressure of the fluid on the solid being equal to ϖ, its components will be evidently $l\varpi$, $m\varpi$, $n\varpi$.

Suppose now the solid to be in motion, and let u', v', w' be the resolved parts of the velocity of the point M of the solid, and ω', ω'', ω''' the angular velocities of the solid. By hypothesis, the forces by which the pressure at any point differs from the normal pressure due to the action of the molecules supposed to be in a state of relative equilibrium about that point are independent of any velocity of translation or rotation. Supposing then linear and angular velocities equal and opposite to those of the solid impressed both on the solid and on the fluid, the former will be for an instant at rest, and we have only to treat the resulting velocities of the fluid as in the first case. Hence $P' = l\varpi + \nu \cdot (u - u')$, &c.; and in Equation 2.8 we may employ the actual velocities u, v, w, since the pressures P, Q, R are independent of any motion of translation and rotation common to the whole fluid. Hence the equations $P' = P$, &c. gives us

$$\left.\begin{aligned}
&l \cdot (\varpi - p) + \nu \cdot (u - u') + \mu \cdot \left\{ 2l \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}x} - \delta \right) + m \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}x} \right) \right. \\
&\left. + n \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}x} \right) \right\} = 0 \,, \\
&m \cdot (\varpi - p) + \nu \cdot (v - v') + \mu \cdot \left\{ l \cdot \left(\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y} \right) + 2m \cdot \left(\frac{\mathrm{d}v}{\mathrm{d}y} - \delta \right) \right. \\
&\left. + n \cdot \left(\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}y} \right) \right\} = 0 \,, \\
&n \cdot (\varpi - p) + \nu \cdot (w - w') + \mu \cdot \left\{ l \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z} \right) + m \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}z} \right) \right. \\
&\left. + 2n \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}z} - \delta \right) \right\} = 0 \,,
\end{aligned}\right\} \tag{2.17}$$

which three equations with Equation 2.15 are those which must be satisfied at the surface of a solid, together with the equation $L = 0$. It will be observed that in the case of a free surface the pressures P', Q', R' are given, whereas in the case of the surface of a solid they are known only by the solution of the problem. But on the other hand the form of the surface of the solid is given, whereas the form of the free surface is known only by the solution of the problem.

Dubuat found by experiment that when the mean velocity of water flowing through a pipe is less than about one inch in a second, the water near the inner surface of the pipe is at rest. If these experiments may be trusted, the conditions to be satisfied in the case of small velocities are those which first occurred to me, and which are included in those just given by supposing $\nu = \infty$.

I have said that when the velocity is not very small the tangential force called into

action by the sliding of water over the inner surface of a pipe varies nearly as the square of the velocity. This fact appears to admit of a natural explanation. When a current of water flows past an obstacle, it produces a resistance varying nearly as the square of the velocity. Now even if the inner surface of a pipe is polished we may suppose that little irregularities exist, forming so many obstacles to the current. Each little protuberance will experience a resistance varying nearly as the square of the velocity, from whence there will result a tangential action of the fluid on the surface of the pipe, which will vary nearly as the square of the velocity; and the same will be true of the equal and opposite reaction of the pipe on the fluid. The tangential force due to this cause will be combined with that by which the fluid close to the pipe is kept at rest when the velocity is sufficiently small.[3]

There remains to be considered the case of two fluids having a common surface. Let u', v', w', μ', δ' denote the quantities belonging to the second fluid corresponding to u, &c. belonging to the first. Together with the two equations $L = 0$ and Equation 2.15 we shall have in this case the equation derived from Equation 2.15

[3]Except in the case of capillary tubes, or, in case the tube be somewhat wider, of excessively slow motions, the main part of the resistance depends upon the formation of eddies. This much appears clear; but the precise way in which the eddies act is less evident. The explanation in the text gives probably the correct account of what takes place in the case of a river flowing over a rough stony bed; but in the case of a pipe of fairly smooth interior surface the minute protuberances would be too small to produce much resistance of the same kind as that contemplated in the paragraph beginning near the foot of page 50.

What actually happens appears to be this. The rolling motion of the fluid belonging to the eddies is continually bringing the more swiftly moving fluid which is found nearer to the centre of the pipe close to the surface. And in consequence the gliding or shifting motion of the fluid in the immediate neighbourhood of the surface in such places is very greatly increased, and with it the tangential pressure.

Thus while in some respects these two classes of resistances are similar, in others they are materially different. As typical examples of the two classes we may take, for the first, that of a polished sphere of glass of some size descending by its weight in deep water; for the second, that of a very long circular glass pipe down which water is flowing. In both cases alike eddies are produced, and the eddies once produced ultimately die away. In both cases alike the internal friction of the fluid, and the friction between the fluid and the solid, are intimately connected with the formation of eddies, and it is by friction that the eddies die away, and the kinetic energy of the mass is converted into molecular kinetic energy, that is, heat. But in the first case the resistance depends mainly on the difference of the pressure p in front and rear, the resultant of the other forces of which the expressions are given in Equation 2.8 being comparatively insignificant, while in the second case it is these latter pressures that we are concerned with, the resultant of the pressure p in the direction of the axis of the tube being practically *nil*, even though the polish of the surface be not mathematically perfect.

Hence if, the motion being what it actually is, the fluidity of the fluid were suddenly to become perfect, the immediate effect on the resistance in the first case would be insignificant, while in the second case the resistance would practically vanish. Of course if the fluidity were to remain perfect, the motion after some time would be very different from what it had been before; but that is not a point under consideration.

Some questions connected with the effect of friction in altering the motion of a nearly perfect fluid will be considered further on in discussing the case of motion given in Art. 55 of a paper *On the Critical Values of the Sums of Periodic Series.*

by putting u', v', w' for u, v, w; or, which comes to the same, we shall have the two former equations with

$$l \cdot (u - u') + m \cdot (v - v') + n \cdot (w - w') = 0\,. \tag{2.18}$$

If we consider the principles on which Equation 2.17 were formed to be applicable to the present case, we shall have six more equations to be satisfied, namely Equation 2.17, and the three equations derived from Equation 2.17 by interchanging the quantities referring to the two fluids, and changing the signs of l, m, n. These equations give the value of ϖ, and leave five equations of condition. If we must suppose $\nu = \infty$, as appears most probable, the six equations above mentioned must be replaced by the six $u' = u$, $v' = v$, $w' = w$, and

$$lp - \mu \cdot f(u, v, w) = lp' - \mu' \cdot f(u', v', w')\,,\ \&\text{c.},$$

$f(u, v, w)$ denoting the coefficient of μ in the first of Equation 2.17. We have here six equations of condition instead of five, but then Equation 2.18 becomes identical. @@@@@@ **7.** The most interesting questions connected with this subject require for their solution a knowledge of the conditions which must be satisfied at the surface of a solid in contact with the fluid, which, except perhaps in case of very small motions, are unknown. It may be well however to give some applications of the preceding equations which are independent of these conditions. Let us then in the first place consider in what manner the transmission of sound in a fluid is affected by the tangential action. To take the simplest case, suppose that no forces act on the fluid, so that the pressure and density are constant in the state of equilibrium, and conceive a series of plane waves to be propagated in the direction of the axis of x, so that $u = f(x, t)$, $v = 0$, $w = 0$. Let $p_\prime$ be the pressure, and $\rho_\prime$ the density of the fluid when it is in equilibrium, and put $p = p_\prime + p'$. Then we have from Equation 2.11 and Equation 2.12, omitting the square of the disturbance,

$$\left.\begin{aligned} \frac{1}{\rho_\prime} \cdot \frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\mathrm{d}u}{\mathrm{d}x} &= 0\,, \\ \rho_\prime \cdot \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\mathrm{d}p'}{\mathrm{d}x} - \frac{4}{3}\mu \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} &= 0\,. \end{aligned}\right\} \tag{2.19}$$

Let $A\Delta\rho$ be the increment of pressure due to a very small increment $\Delta\rho$ of density, the temperature being unaltered, and let m be the ratio of the specific heat of the fluid when the pressure is constant to its specific heat when the volume is constant; then the relation between p' and ρ will be

$$p' = mA \cdot (\rho - \rho_\prime)\,. \tag{2.20}$$

Eliminating p' and ρ from Equation 2.19 and Equation 2.20 we get

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - mA \cdot \frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{4\mu}{3\rho_\prime} \cdot \frac{\mathrm{d}^3u}{\mathrm{d}t\,\mathrm{d}x^2} = 0\,.$$

To obtain a particular solution of this equation, let

$$u = \phi(t) \cdot \cos\left(\frac{2\pi x}{\lambda}\right) + \psi(t) \cdot \sin\left(\frac{2\pi x}{\lambda}\right)\,.$$

Substituting in the above equation, we see that $\phi(t)$ and $\psi(t)$ must satisfy the same equation, namely,

$$\frac{\mathrm{d}^2\phi(t)}{\mathrm{d}t^2} + \frac{4\pi^2}{\lambda^2} \cdot mA \cdot \phi(t) + \frac{16\pi^2\mu}{3\lambda^2\rho_\prime} \cdot \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = 0\,,$$

the integral of which is

$$\phi(t) = e^{-ct} \cdot \left[C \cdot \cos\left(\frac{2\pi bt}{\lambda}\right) + C' \cdot \sin\left(\frac{2\pi bt}{\lambda}\right)\right]\,,$$

where

$$c = \frac{8\pi^2\mu}{3\lambda^2\rho_\prime}\,,$$
$$b^2 = mA - \frac{16\pi^2\mu^2}{9\lambda^2\rho_\prime^2}\,,$$

C and C' being arbitrary constants. Taking the same expression with different arbitrary constants for $\psi(t)$, replacing products of sines and cosines by sums and differences, and combining the resulting sines and cosines two and two, we see that the resulting value of u represents two series of waves propagated in opposite directions. Considering only those waves which are propagated in the positive direction of x, we have

$$u = C_1 \cdot e^{-ct} \cdot \cos\left\{\frac{2\pi}{\lambda} \cdot (bt - x) + C_2\right\}\,. \tag{2.21}$$

We see then that the effect of the tangential force is to make the intensity of the sound diminish as the time increases, and to render the velocity of propagation less than what it would otherwise be. Both effects are greater for high, than for low notes; but the former depends on the first power of μ, while the latter depends only on μ^2. It appears from the experiments of M. Biot, made on empty water pipes in Paris, that the velocity of propagation of sound is sensibly the same whatever be its pitch. Hence it is necessary to suppose that for air $\frac{\mu^2}{\lambda^2\rho_\prime^2}$ is insensible compared

with A or $\frac{p_\prime}{\rho_\prime}$. I am not aware of any similar experiments made on water, but the ratio of $\left(\frac{\mu}{\lambda\rho_\prime}\right)^2$ to A would probably be insensible for water also. The diminution of intensity as the time increases is, in the case of plane waves, due *entirely* to friction; but as we do not possess any means of measuring the intensity of sound, the theory cannot be tested, nor the numerical value of μ be determined, in this way.

The velocity of sound in air, deduced from the note given by a known tube, is sensibly less than that determined by direct observation. POISSON thought that this might be due to the retardation of the air by friction against the sides of the tube. But from the above investigation it seems unlikely that the effect produced by that cause would be sensible.

Equation 2.21 may be considered as expressing in all cases the effect of friction; for we may represent an arbitrary disturbance of the medium as the aggregate of series of plane waves propagated in all directions.

8. Let us now consider the motion of a mass of uniform inelastic fluid comprised between two cylinders having a common axis, the cylinders revolving uniformly about their axis, and the fluid being supposed to have attained its permanent state of motion. Let the axis of the cylinders be taken for that of z, and let q be the actual velocity of any particle, so that $u = -q \cdot \sin(\theta)$, $v = q \cdot \cos(\theta)$, $w = 0$, r and θ being polar co-ordinates in a plane parallel to xy. Observing that

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 f}{\mathrm{d}y^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r} \cdot \frac{\mathrm{d}f}{\mathrm{d}r} + \frac{1}{r^2} \cdot \frac{\mathrm{d}^2 f}{\mathrm{d}\theta^2}\,,$$

where f is any function of x and y, and that $\frac{\mathrm{d}p}{\mathrm{d}\theta} = 0$, we have from Equation 2.13, supposing after differentiation that the axis of x coincides with the radius vector of the point considered, and omitting the forces, and the part of the pressure due to them,

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}r} - \rho \cdot \frac{q^2}{r} &= 0\,, \\ \frac{\mathrm{d}^2 q}{\mathrm{d}r^2} + \frac{1}{r} \cdot \frac{\mathrm{d}q}{\mathrm{d}r} - \frac{q}{r^2} &= 0\,, \end{aligned} \tag{2.22}$$

and the equation of continuity is satisfied identically.

The integral of Equation 2.22 is

$$q = \frac{C}{r} + C'r\,.$$

If a is the radius of the inner, and b that of the outer cylinder, and if q_1, q_2 are the velocities of points close to these cylinders respectively, we must have $q = q_1$, when

$r = a$, and $q = q_2$, when $r = b$, whence

$$q = \frac{1}{(b^2 - a^2)} \cdot \left\{ (bq_1 - aq_2) \cdot \frac{ab}{r} + (bq_2 - aq_1) \cdot r \right\} \tag{2.23}$$

If the fluid is infinitely extended, it follows that $b = \infty$, and

$$\frac{q}{q_1} = \frac{a}{r}.$$

These cases of motion were considered by NEWTON, (*Principia*, Lib. II. Prop. 51.). The hypothesis which I have made agrees in this case with his, but he arrives at the result that the velocity is constant, not, that it varies inversely as the distance. This arises from his having taken, as the condition of there being no acceleration or retardation of the motion of an annulus, that the force tending to turn it in one direction must be equal to that tending to turn it in the opposite, whereas the true condition is that the moment of the force tending to turn it one way must be equal to the moment of the force tending to turn it the other. Of course, making this alteration, it is easy to arrive at the above result by NEWTON's reasoning. The error just mentioned vitiates the result of Prop. 52. It may be shewn from the general equations that in this case a permanent motion in annuli is impossible, and that, whatever may be the law of friction between the solid sphere and the fluid. Hence it appears that it is necessary to suppose that the particles move in planes passing through the axis of rotation, while they at the same time move round it. In fact, it is easy to see that from the excess of centrifugal force in the neighbourhood of the equator of the revolving sphere the particles in that part will recede from the sphere, and approach it again in the neighbourhood of the poles, and this circulating motion will be combined with a motion about the axis. If however we leave the centrifugal force out of consideration, as NEWTON has done, the motion in annuli becomes possible, but the solution is different from NEWTON's, as might have been expected.

The case of motion considered in this article may perhaps admit of being compared with experiment, without knowing the conditions which must be satisfied at the surface of a solid. A hollow, and a solid cylinder might be so mounted as to admit of being turned with different uniform angular velocities round their common axis, which is supposed to be vertical. If both cylinders are turned, they ought to be turned in opposite directions, if only one, it ought to be the outer one; for if the inner were made to revolve too fast, the fluid near it would have a tendency to fly outwards in consequence of the centrifugal force, and eddies would be produced. As long as the angular velocities are not great, so that the surface of the liquid is very nearly plane, it is not of much importance that the fluid is there terminated; for the conditions which must be satisfied at a free surface are satisfied for any section of the fluid made by a horizontal plane, so long as the motion about that section is supposed to be the same as it would be if the cylinders were infinite. The

principal difficulty would probably be to measure accurately the time of revolution, and distance from the axis, of the different annuli. This would probably be best done by observing motes in the fluid. It might be possible also to discover in this way the conditions to be satisfied at the surface of the cylinders; or at least a law might be suggested, which could be afterwards compared more accurately with experiment by means of the discharge of pipes and canals.
If the rotations of the cylinders are in opposite directions, there will be a certain distance from the axis at which the fluid will not revolve at all. Writing $-q_1$ for q_1 in Equation 2.23, we have for this distance

$$\sqrt{\frac{ab \cdot (bq_1 + aq_2)}{(bq_2 + aq_1)}} .$$

9. Although the discharge of a liquid through a long straight pipe or canal, under given circumstances, cannot be calculated without knowing the conditions to be satisfied at the surface of contact of the fluid and solid, it may be well to go a certain way towards the solution.
Let the axis of z be parallel to the generating lines of the pipe or canal, and inclined at an angle α to the horizon; let the plane yz be vertical, and let y and z be measured downwards. The motion being uniform, we shall have $u = 0$, $v = 0$, $w = f(x, y)$, and we have from Equation 2.13

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}x} &= 0 \,, \\
\frac{\mathrm{d}p}{\mathrm{d}y} &= g\rho \cdot \cos(\alpha) \,, \\
\frac{\mathrm{d}p}{\mathrm{d}z} &= g\rho \cdot \sin(\alpha) + \mu \cdot \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 w}{\mathrm{d}y^2} \right) .
\end{aligned}$$

In the case of a canal $\frac{\mathrm{d}p}{\mathrm{d}z} = 0$; and the calculation of the motion in a pipe may always be reduced to that of the motion in the same pipe when $\frac{\mathrm{d}p}{\mathrm{d}z}$ is supposed to be zero, as may be shewn by reasoning similar to Dubuat's. Moreover the motion in a canal is a particular case of the motion in a pipe. For consider a pipe for which $\frac{\mathrm{d}p}{\mathrm{d}z} = 0$, and which is divided symmetrically by the plane xz. From the symmetry of the motion, it is clear that we must have $\frac{\mathrm{d}w}{\mathrm{d}y} = 0$ when $z = 0$; but this is precisely the condition which would have to be satisfied if the fluid had a free surface coinciding with the plane xz; hence we may suppose the upper half of the fluid removed, without affecting the motion of the rest, and thus we pass to the case of a canal. Hence it is the same thing to determine the motion in a canal, as to determine that in the pipe formed by completing the canal symmetrically with respect to the surface of the fluid.

We have then, to determine the motion, the equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 w}{\mathrm{d}y^2} + \frac{g\rho \cdot \sin(\alpha)}{\mu} = 0\,.$$

In the case of a rectangular pipe, it would not be difficult to express the value of w at any point in terms of its values at the several points of the perimeter of a section of the pipe. In the case of a cylindrical pipe the solution is extremely easy: for if we take the axis of the pipe for that of z, and take polar co-ordinates r, θ in a plane parallel to xy, and observe that $\frac{\mathrm{d}w}{\mathrm{d}\theta} = 0$, since the motion is supposed to be symmetrical with respect to the axis, the above equation becomes

$$\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r} \cdot \frac{\mathrm{d}w}{\mathrm{d}r} + \frac{g\rho \cdot \sin(\alpha)}{\mu} = 0\,.$$

Let a be the radius of the pipe, and U the velocity of the fluid close to the surface; then, integrating the above equation, and determining the arbitrary constants by the conditions that w shall be finite when $r = 0$, and $w = U$ when $r = a$, we have

$$w = \frac{g\rho \cdot \sin(\alpha)}{4\mu} \cdot \left(a^2 - r^2\right) + U\,.$$

3. Objections to Lagrange's proof of the theorem that if $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential at any one instant it is always so, the pressure being supposed equal in all directions. Principles of M. Cauchy's proof. A new proof of the theorem. A physical interpretation of the circumstance of the above expression being an exact differential

10. The proof of this theorem given by Lagrange depends on the legitimacy of supposing u, v and w capable of expansion according to positive integral powers of t, for a sufficiently small finite value of t. It is clear that the expansion cannot contain negative powers of t, since u, v and w are supposed to be finite when $t = 0$; but it may be objected to LAGRANGE's proof that there are functions of t of which the expansion contains fractional powers of t, and that we do not know but that u, v and w may be such functions. This objection has been considered by Mr. POWER[1], who has shewn that the theorem is true if we suppose u, v and w capable of expansion according to any powers of t. Still the proof remains unsatisfactory, in fact inconclusive, for these are functions of t, (for instance $e^{-\frac{1}{t^2}}$,

[1] *Cambridge Philosophical Transactions*, Vol, VII. Part 3., https://www.biodiversitylibrary.org/page/2398129

$t \cdot \ln(t)$,) which do not admit of expansion according to powers of t, integral or fractional, and we do not know but that u, v and w may be functions of this nature. I do not here mention the proof which POISSON has given of the theorem in his *Traité de Méchanique*, because it appears to me liable to an objection to which I shall presently have occasion to refer: in fact, POISSON himself did not think the theorem generally true.

It is remarkable that Mr. POWER's proof, if it were legitimate, would establish the theorem even when account is taken of the variation of pressure in different directions, according to the theory explained in chapter 2, if we suppose that $\frac{\mathrm{d}\mu}{\mathrm{d}p} = 0$. To shew this we have only got to treat Equation 2.12 as Mr. POWER has treated the three equations of fluid motion formed on the ordinary hypothesis. Yet in this case the theorem is evidently untrue. Thus, conceive a mass of fluid which is bounded by a solid plane coinciding with the plane yz, and which extends infinitely in every direction on the positive side of the axis of x, and suppose the fluid at first to be at rest. Suppose now the solid plane to be moved in any manner parallel to the axis of y; then, unless the solid plane exerts no tangential force on the fluid, (and we may suppose that it does exert some,) it is clear that at a given time we shall have $u = 0$, $v = f(x)$, $w = 0$, and therefore $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ will not be an exact differential. It will be interesting then to examine in this case the nature of the function of t which expresses the value of v.

Supposing X, Y, Z to be zero in Equation 2.12, and observing that in the case considered we have $\frac{\mathrm{d}p}{\mathrm{d}y} = 0$, we get

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mu}{\rho} \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\,. \tag{3.1}$$

Differentiating this equation $n - 1$ times with respect to t, we easily get

$$\frac{\mathrm{d}^n v}{\mathrm{d}t^n} = \left(\frac{\mu}{\rho}\right)^n \cdot \frac{\mathrm{d}^{2n} v}{\mathrm{d}x^{2n}}\ :$$

but when $t = 0$, $v = 0$ when $x > 0$, and therefore for a given value of x all the differential coefficients of v with respect to t are zero. Hence for indefinitely small values of t the value of v at a given point increases more slowly than if it varied ultimately as any power of t, however great; hence v cannot be expanded in a series according to powers of t. This result is independent of the condition to be satisfied at the surface of the solid plane.

I think what has just been proved shews clearly that LAGRANGE's proof of the theorem considered, even with Mr. POWER's improvement of it, is inadmissible.

11. The theorem is however true, and a proof of it has been given by M. CAUCHY[2], which appears to me perfectly free from objection, and which is very simple in principle, although it depends on rather long equations. M. CAUCHY first eliminates p from the three equations of motion by means of the conditions that $\frac{\mathrm{d}^2 p}{\mathrm{d}x\,\mathrm{d}y} = \frac{\mathrm{d}^2 p}{\mathrm{d}y\,\mathrm{d}x}$, &c., he then changes the independent variables from x, y, z, t to a, b, c, t, where a, b, c are the initial co-ordinates of the particles. The three transformed equations admit each of being once integrated with respect to t; and determining the arbitrary functions of a, b, c by the initial values of u, v and w, the three integrals have the form

$$\omega'_0 = F\omega' + G\omega'' + H\omega''' \text{ , \&c.,}$$

ω', ω'' and ω''' denoting here the same as in Art. 2, page 74, and ω'_0, &c. denoting the initial values of ω', &c. for the same particle. Solving the above equations with respect to ω', ω'' and ω''', the resulting equations are

$$\omega' = \frac{1}{S} \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}a}\omega'_0 + \frac{\mathrm{d}x}{\mathrm{d}b}\omega''_0 + \frac{\mathrm{d}x}{\mathrm{d}c}\omega'''_0 \right) \text{ , \&c.,}$$

where S is a function of the differential coefficients of x, y and z with respect to a, b and c, which by the condition of continuity is shewn to be equal to $\frac{\rho}{\rho_0}$, ρ_0 being the initial density about the particle whose density at the time considered is ρ. Since $\frac{\mathrm{d}x}{\mathrm{d}a}$, &c. are finite, (for to suppose them infinite would be equivalent to supposing a discontinuity to exist in the fluid,) it follows at once from the preceding equations that if $\omega'_0 = 0$, $\omega''_0 = 0$, $\omega'''_0 = 0$, that is if $u_0\,\mathrm{d}a + v_0\,\mathrm{d}b + w_0\,\mathrm{d}c$ be an exact differential, either for the whole fluid or for any portion of it, then shall $\omega' = 0$, $\omega'' = 0$, $\omega''' = 0$, i.e. $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ will be an exact differential, at any subsequent time, either for the whole mass or for the above portion of it.

[2] *Mémoire sur la Théorie des Ondes*, in the first volume of the *Mémoires des Savants étrangers à l'Académie de Paris*, `https://gallica.bnf.fr/ark:/12148/bpt6k33065?rk=21459;2`. M. CAUCHY has not had occasion to enunciate the theorem, but it is contained in his equations (16). This equation may be obtained in the same manner in the more general case in which p is supposed to be a function of ρ.

12. It is not from seeing the smallest flaw in M. CAUCHY's proof that I propose a new one, but because it is well to view the subject in different lights, and because the proof which I am about to give does not require such long equations. It will be necessary in the first place to prove the following lemma.

Lemma. *If ω_1, ω_2, ..., ω_n are n functions of t, which satisfy the n differential equations*

$$\left.\begin{aligned} \frac{\mathrm{d}\omega_1}{\mathrm{d}t} &= P_1\omega_1 + Q_1\omega_2 + \cdots + V_1\omega_n\,, \\ &\vdots \\ \frac{\mathrm{d}\omega_n}{\mathrm{d}t} &= P_n\omega_1 + Q_n\omega_2 + \cdots + V_n\omega_n\,, \end{aligned}\right\} \tag{3.2}$$

where P_1, Q_1, ..., V_n may be functions of t, ω_1, ω_2, ..., ω_n, and if when $\omega_1 = 0$, $\omega_2 = 0$, ..., $\omega_n = 0$, none of the quantities P_1, Q_1, ..., V_n is infinite for any value of t from 0 to T, and if ω_1, ω_2, ..., ω_n are each zero when $t = 0$, then shall each of these quantities remain zero for all values of t from 0 to T.

Proof. Let τ be a finite value of t, then by hypothesis τ may be taken so small that the values of ω_1, ω_2, ..., ω_n are sufficiently small to exclude all values which might render any one of the quantities P_1, Q_1, ..., V_n infinite. Let L be a superior limit to the numerical values of the several quantities P_1, Q_1, ..., V_n for all values of t from 0 to τ; then it is evident that ω_1, ω_2, ..., ω_n cannot increase faster than if they satisfied the equations

$$\left.\begin{aligned} \frac{\mathrm{d}\omega_1}{\mathrm{d}t} &= L\left(\omega_1 + \omega_2 + \cdots + \omega_n\right)\,, \\ &\vdots \\ \frac{\mathrm{d}\omega_n}{\mathrm{d}t} &= L\left(\omega_1 + \omega_2 + \cdots + \omega_n\right)\,, \end{aligned}\right\} \tag{3.3}$$

vanishing in this case also when $t = 0$. But if $(\omega_1 + \omega_2 + \cdots + \omega_n) = \Omega$, we have by adding together the above equations

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = nL\Omega\ :$$

if now Ω be not equal to zero, dividing this equation by Ω and integrating, we have

$$\Omega = C \cdot e^{nLt}\,;$$

but no value of C different from zero will allow Ω to vanish when $t = 0$, whereas

by hypothesis it does vanish; hence $\Omega = 0$; but Ω is the sum of n quantities which evidently cannot be negative, and therefore each of these must be zero. Since then $\omega_1, \omega_2, \ldots, \omega_n$ would have to be equal to zero for all values of t from 0 to τ even if they satisfied Equation 3.3, they must *à fortiori* be equal to zero in the actual case, since they satisfy Equation 3.2. Hence there is no value of t from 0 to T at which any one of the quantities $\omega_1, \omega_2, \ldots, \omega_n$ can begin to differ from zero, and therefore these quantities must remain equal to zero for all values of t from 0 to T. □

This lemma might be extended to the case in which $n = \infty$, with certain restrictions as to the convergency of the series. We may also, instead of the integers 1, 2, ... n, have a continuous variable α which varies from 0 to a, so that ω is a function of the independent variables α and t, satisfying the differential equation

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \int_0^\alpha \psi(\alpha, \omega, t)\omega\,\mathrm{d}\alpha\,,$$

where $\psi(\alpha, 0, t)$ does not become infinite for any value of α from 0 to a combined with any value of t from 0 to T. It may be shewn, just as before, that if $\omega = 0$ when $t = 0$ for all values of a from 0 to a, then must $\omega = 0$ for all values of t from 0 to T. The proposition might be further extended to the case in which $a = \infty$, with a certain restriction as to the convergency of the integral, but Equation 3.2 are already more general than I shall have occasion to employ.

It appears to me to be sometimes assumed as a principle that two variables, functions of another, t, are proved to be equal for all values of t when it is shewn that they are equal for a certain value of t, and that whenever they are equal for the same value of t their increments for the same increment of t are *ultimately* equal. But according to this principle, if two curves could be shewn always to touch when they meet they must always coincide, a conclusion manifestly false. I confess I cannot see that NEWTON in his *Principia*, Lib. I. Prop. 40 has proved more than that if the velocities of the two bodies are equal at equal distances, the increments of those velocities for equal increments of the distances are ultimately equal: at least something additional seems required to put the proof quite out of the reach of objection. Again it is usual to speak of the condition, that the motion of a particle of fluid in contact with the surface of a solid at rest is tangential to the surface, as the same thing as the condition that the particle shall always remain in contact with the surface. That it is the same thing that might be shewn by means of the lemma in this article, supposing the motion to be continuous; but independently of proof I do not see why a particle should not move in a curve not coinciding with the surface, but touching it where it meets it. The same remark will apply to the condition that a particle which at one instant lies in a free surface, or is in contact with a solid, shall ultimately lie in the free surface, or be in contact with the solid, at the consecutive instant. I refer here to the more general case in which the solid is at rest or in motion. For similar reasons POISSON's proof of the Hydrodynamical

theorem which forms the principal subject of this section has always appeared to me unsatisfactory, in fact far less satisfactory than LAGRANGE's. I may add that POISSON's proof, as well as LAGRANGE's, would apply to the case in which friction is taken into account, in which case the theorem is not true.

13. Supposing ρ to be a function of p, $\frac{1}{f'(p)}$, the ordinary equations of Hydrodynamics are

$$\left.\begin{aligned} \frac{\mathrm{d}f(p)}{\mathrm{d}x} &= X - \frac{\mathrm{D}u}{\mathrm{D}t}\,, \\ \frac{\mathrm{d}f(p)}{\mathrm{d}y} &= Y - \frac{\mathrm{D}v}{\mathrm{D}t}\,, \\ \frac{\mathrm{d}f(p)}{\mathrm{d}z} &= Z - \frac{\mathrm{D}w}{\mathrm{D}t}\,. \end{aligned}\right\} \tag{3.4}$$

The forces X, Y, Z will here be supposed to be such that $X\,\mathrm{d}x + Y\,\mathrm{d}y + Z\,\mathrm{d}z$ is an exact differential, this being the case for any forces emanating from centres, and varying as any functions of the distances. Differentiating the first of Equation 3.4 with respect to y, and the second with respect to x, subtracting, putting for $\frac{\mathrm{D}u}{\mathrm{D}t}$ and $\frac{\mathrm{D}v}{\mathrm{D}t}$ their values, adding and subtracting $\frac{\mathrm{d}u}{\mathrm{d}z}\frac{\mathrm{d}v}{\mathrm{d}z}$, and employing the notation of Art. 2, page 74, we obtain

$$\frac{\mathrm{D}\omega'''}{\mathrm{D}t} = \frac{\mathrm{d}u}{\mathrm{d}z}\omega' + \frac{\mathrm{d}v}{\mathrm{d}z}\omega'' - \left(\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y}\right)\omega'''\,. \tag{3.5}$$

By treating the first and third, and then the second and third of Equation 3.4 in the same manner, we should obtain two more equations, which may be got at once from that which has just been found by interchanging the requisite quantities. Now for points in the interior of the mass the differential coefficients $\frac{\mathrm{d}u}{\mathrm{d}z}$, &c. will not be infinite, on account of the continuity of the motion, and therefore the three equations just obtained are a particular case of Equation 3.2. If then $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential for any portion of the fluid when $t = 0$, that is, if ω', ω'' and ω''' are each zero when $t = 0$, it follows from the lemma of the last article (see page 104) that ω', ω'' and ω''' will be zero for any value of t, and therefore $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ will always remain an exact differential. It will be observed that it is for the same portion of fluid, not for the fluid occupying the same portion of space, that this is true, since Equation 3.5, &c. contain the differential coefficients $\frac{\mathrm{D}\omega'}{\mathrm{D}t}$ &c., and not $\frac{\mathrm{d}\omega'}{\mathrm{d}t}$, &c.

14. The circumstance of $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ being an exact differential admits of a physical interpretation which may be noticed, as it is well to view a subject of this nature in different lights.

Conceive an indefinitely small element of a fluid in motion to become suddenly solidified, and the fluid about it to be suddenly destroyed; let the form of the element be so taken that the resulting solid shall be that which is the simplest with

respect to rotatory motion, namely, that which has its three principal moments about axes passing through the centre of gravity equal to each other, and therefore every axis passing through that point is a principal axis, and let us enquire what will be the linear and angular motion of this element just after solidification.
By the instantaneous solidification, velocities will be suddenly generated or destroyed in the different portions of the element, and a set of mutual impulsive forces will be called into action. Let x, y, z be the co-ordinates of the centre of gravity G of the element at the instant of solidification, $x + x'$, $y + y'$, $z + z'$ those of any other point in it. Let u, v, w be the velocities of G along the three axes just before solidification, u', v', w' the relative velocities of the point whose relative co-ordinates are x', y', z'. Let $\overline{u}$, $\overline{v}$, $\overline{w}$ be the velocities of G, $u_\prime$, $v_\prime$, $w_\prime$ the relative velocities of the point above mentioned, and ω', ω'', ω''' the angular velocities just after solidification. Since all the impulsive forces are internal, we have

$$\overline{u} = u\,, \quad \overline{v} = v\,, \quad \overline{w} = w\,.$$

We have also, by the principle of conservation of areas,

$$\begin{aligned}
\sum m\,\{y' \cdot (w_\prime - w') - z' \cdot (v_\prime - v')\} &= 0\,,\\
\sum m\,\{z' \cdot (u_\prime - u') - x' \cdot (w_\prime - w')\} &= 0\,,\\
\sum m\,\{x' \cdot (v_\prime - v') - y' \cdot (u_\prime - u')\} &= 0\,,
\end{aligned}$$

m denoting an element of the mass of the element considered. But $u_\prime = \omega'' z' - \omega''' y'$, u' is ultimately equal to

$$\frac{\mathrm{d}u}{\mathrm{d}x}x' + \frac{\mathrm{d}u}{\mathrm{d}y}y' + \frac{\mathrm{d}u}{\mathrm{d}z}z'$$

and similar expressions hold good for the other quantities. Substituting in the above equations, and observing that $\sum my'z' = \sum mz'x' = \sum mx'y' = 0$, and $\sum mx'^2 + \sum my'^2 + \sum mz'^2$ we have

$$\begin{aligned}
\omega' &= \frac{1}{2} \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}y} - \frac{\mathrm{d}v}{\mathrm{d}z}\right)\,,\\
\omega'' &= \frac{1}{2} \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}z} - \frac{\mathrm{d}w}{\mathrm{d}x}\right)\,,\\
\omega''' &= \frac{1}{2} \cdot \left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}y}\right)\,.
\end{aligned}$$

We see then that an indefinitely small element of the fluid, of which the three principal moments about the centre of gravity are equal, if suddenly solidified and detached from the rest of the fluid will begin to move with a motion simply of translation, which may however vanish, or a motion of translation combined with

one of rotation, according as $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is, or is not an exact differential, and in the latter case the angular velocities will be the same as in Art. 2, page 74.

The principle which forms the subject of this section might be proved, at least in the case of a homogeneous incompressible fluid, by considering the change in the motion of a spherical element of the fluid in the indefinitely small time $\mathrm{d}t$. This method of proving the principle would shew distinctly its intimate connexion with the hypothesis of normal pressure, or the equivalent hypothesis of the equality of pressure in all directions, since the proof depends on the impossibility of an angular velocity being generated in the element in the indefinitely small time $\mathrm{d}t$ by the pressure of the surrounding fluid, inasmuch as the direction of the pressure at any point of the surface ultimately passes through the centre of the sphere. The proof I speak of is however less simple than the one already given, and would lead me too far from my subject.

4. Application of a method analogous to that of chapter 2 to the determination of the equations of equilibrium and motion of elastic solids

15. All solid bodies are more or less elastic, as is shewn by the capability they possess of transmitting sound, and vibratory motions in general. The solids considered in this section are supposed to be homogeneous and uncrystallized, so that when in their natural state the average arrangement of their particles is the same at one point as at another, and the same in one direction as in another.[1] The natural state will be taken to be that in which no forces act on them, from which it may be shewn that the pressure in the interior is zero at all points and in all directions, neglecting the small pressure depending on attractions of the nature of capillary attraction.

Let x, y, z be the co-ordinates of any point P in the solid considered when in its natural state, α, β, γ the increments of those co-ordinates at the time considered, whether the body be in a state of constrained equilibrium or of motion. It will be supposed that α, β, and γ are so small that their squares and products may be neglected. All the theorems proved in Art. 2., page 74 with reference to linear and angular velocities will be true here with reference to linear and angular displacements, since these two sets of quantities are resolved according to the same laws, as long as the angular displacements are supposed to be very small. Thus, the most general displacement of a very small element of the solid consists of a displacement of translation, an angular displacement, and three displacements of extension in the direction of three rectangular axes, which may be called in this case, with more propriety than in the former, *axes of extension*. The three displacements of extension may be resolved into two displacements of shifting, each in two dimensions, and a displacement of uniform dilatation, positive or negative. The pressures about the element considered will depend on the displacements of

[1] In modern notation this reads: "The solids considered in this section are supposed to be *isotropic*". *Isotropy* (from Ancient Greek ísos "equal", and trópos "turn, way") is uniformity in all orientations.

extension only; there may also, in the case of motion, be a small part depending on the relative velocities, but this part may be neglected, unless we have occasion to consider the effect of the internal friction in causing the vibrations of solid bodies to subside. It has been shewn (Art. 7., page 94) that the effect of this cause is insensible in the case of sound propagated through air; and there is no reason to suppose it greater in the case of solids than in the case of fluids, but rather the contrary. The capability which solids possess of being put into a state of isochronous vibration shews that the pressures called into action by small displacements depend on homogeneous functions of those displacements of one dimension. I shall suppose moreover, according to the general principle of the superposition of small quantities, that the pressures due to different displacements are superimposed, and consequently that the pressures are linear functions of the displacements. Since squares of α, β, and γ are neglected, these pressures may be referred to a unit of surface in the natural state or after displacement indifferently, and a pressure which is normal to any surface after displacement may be regarded as normal to the original position of that surface. Let $-A\delta$ be the pressure corresponding to a uniform linear dilatation δ when the solid is in equilibrium, and suppose that it becomes $-mA\delta$, in consequence of the heat developed, when the solid is in a state of rapid vibration. Suppose also that a displacement of shifting parallel to the plane xy, for which $\alpha = k \cdot x$, $\beta = -k \cdot y$, $\gamma = 0$, calls into action a pressure $-Bk$ on a plane perpendicular to the axis of x, and a pressure Bk on a plane perpendicular to that of y; the pressures on these planes being equal and of opposite signs, that on a plane perpendicular to the axis of z being zero, and the tangential forces on those planes being zero, for the same reasons as in chapter 2. It may also be shewn as before that it is necessary to suppose B positive, in order that the equilibrium of the solid medium may be stable, and it is easy to see that the same must be the case with A for the same reason.

It is clear that we shall obtain the expressions for the pressures from those already found for the case of a fluid by merely putting α, β, γ, B for u, v, w, μ and $-A\delta$ or $-mA\delta$ for p, according as we are considering the case of equilibrium or of vibratory motion, the body being in the latter case supposed to be constrained only in so far as depends on the motion.

For the case of equilibrium then we have from Equation 2.8

$$\left.\begin{aligned} P_1 &= -A\delta - 2B\cdot\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x} - \delta\right), \\ P_2 &= -A\delta - 2B\cdot\left(\frac{\mathrm{d}\beta}{\mathrm{d}y} - \delta\right), \\ P_3 &= -A\delta - 2B\cdot\left(\frac{\mathrm{d}\gamma}{\mathrm{d}z} - \delta\right), \\ T_1 &= -B\cdot\left(\frac{\mathrm{d}\beta}{\mathrm{d}z} + \frac{\mathrm{d}\gamma}{\mathrm{d}y}\right), \\ T_2 &= -B\cdot\left(\frac{\mathrm{d}\gamma}{\mathrm{d}x} + \frac{\mathrm{d}\alpha}{\mathrm{d}z}\right), \\ T_3 &= -B\cdot\left(\frac{\mathrm{d}\alpha}{\mathrm{d}y} + \frac{\mathrm{d}\beta}{\mathrm{d}x}\right), \end{aligned}\right\} \tag{4.1}$$

δ being here

$$\frac{1}{3}\cdot\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x} + \frac{\mathrm{d}\beta}{\mathrm{d}y} + \frac{\mathrm{d}\gamma}{\mathrm{d}z}\right);$$

and the equations of equilibrium will be obtained from Equation 2.12 by putting $\frac{\mathrm{D}u}{\mathrm{D}t} = 0$, $p = -A\delta$, making the same substitution as before for u, v, w and μ. We have therefore, for the equations of equilibrium,

$$\rho\cdot X + \frac{1}{3}\cdot(A+B)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x} + \frac{\mathrm{d}\beta}{\mathrm{d}y} + \frac{\mathrm{d}\gamma}{\mathrm{d}z}\right) + B\cdot\left(\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2} + \frac{\mathrm{d}^2\alpha}{\mathrm{d}y^2} + \frac{\mathrm{d}^2\alpha}{\mathrm{d}z^2}\right) = 0\,,\ \text{\&c.} \tag{4.2}$$

In the case of a vibratory motion, when the body is in its natural state except so far as depends on the motion, we have from Equation 2.8

$$\left.\begin{aligned} P_1 &= -mA\delta - 2B\cdot\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x} - \delta\right), \\ T_1 &= -B\cdot\left(\frac{\mathrm{d}\beta}{\mathrm{d}z} + \frac{\mathrm{d}\gamma}{\mathrm{d}y}\right),\ \text{\&c.}, \end{aligned}\right\} \tag{4.3}$$

and the equations of motion will be derived from Equation 2.12 as before, only $\frac{\mathrm{D}u}{\mathrm{D}t}$ &c. must be replaced by $\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}$ &c., and X, Y, Z put equal to zero. The equations

of motion, then, are

$$\left.\begin{aligned}
\rho\cdot\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} &= \frac{1}{3}\cdot(mA+B)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x}+\frac{\mathrm{d}\beta}{\mathrm{d}y}+\frac{\mathrm{d}\gamma}{\mathrm{d}z}\right)\\
&\quad + B\cdot\left(\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2}+\frac{\mathrm{d}^2\alpha}{\mathrm{d}y^2}+\frac{\mathrm{d}^2\alpha}{\mathrm{d}z^2}\right),\\
\rho\cdot\frac{\mathrm{d}^2\beta}{\mathrm{d}t^2} &= \frac{1}{3}\cdot(mA+B)\cdot\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x}+\frac{\mathrm{d}\beta}{\mathrm{d}y}+\frac{\mathrm{d}\gamma}{\mathrm{d}z}\right)\\
&\quad + B\cdot\left(\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2}+\frac{\mathrm{d}^2\beta}{\mathrm{d}y^2}+\frac{\mathrm{d}^2\beta}{\mathrm{d}z^2}\right),\\
\rho\cdot\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} &= \frac{1}{3}\cdot(mA+B)\cdot\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x}+\frac{\mathrm{d}\beta}{\mathrm{d}y}+\frac{\mathrm{d}\gamma}{\mathrm{d}z}\right)\\
&\quad + B\cdot\left(\frac{\mathrm{d}^2\gamma}{\mathrm{d}x^2}+\frac{\mathrm{d}^2\gamma}{\mathrm{d}y^2}+\frac{\mathrm{d}^2\gamma}{\mathrm{d}z^2}\right),
\end{aligned}\right\} \tag{4.4}$$

16. The conditions to be satisfied at the surface of the solid may be easily deduced from the analogous conditions in the case of a fluid with a free surface, only it will be necessary to replace the normal pressure Π by an oblique pressure, of which the components will be denoted by X_1, Y_1, Z_1. We have then, making the necessary changes in the quantities involved in Equation 2.14,

$$\begin{aligned}
X_1 + lA\delta + B\cdot\left\{2l\cdot\frac{\mathrm{d}\alpha}{\mathrm{d}x}+m\cdot\left(\frac{\mathrm{d}\alpha}{\mathrm{d}y}+\frac{\mathrm{d}\beta}{\mathrm{d}x}\right)+n\cdot\left(\frac{\mathrm{d}\alpha}{\mathrm{d}z}+\frac{\mathrm{d}\gamma}{\mathrm{d}x}\right)\right\} &= 0\,,\\
Y_1 + mA\delta + B\cdot\left\{l\cdot\left(\frac{\mathrm{d}\beta}{\mathrm{d}z}+\frac{\mathrm{d}\gamma}{\mathrm{d}y}\right)+2m\cdot\frac{\mathrm{d}\beta}{\mathrm{d}y}+n\cdot\left(\frac{\mathrm{d}\beta}{\mathrm{d}x}+\frac{\mathrm{d}\alpha}{\mathrm{d}y}\right)\right\} &= 0\,,\\
Z_1 + nA\delta + B\cdot\left\{l\cdot\left(\frac{\mathrm{d}\gamma}{\mathrm{d}x}+\frac{\mathrm{d}\alpha}{\mathrm{d}z}\right)+m\cdot\left(\frac{\mathrm{d}\gamma}{\mathrm{d}y}+\frac{\mathrm{d}\beta}{\mathrm{d}z}\right)+2n\cdot\frac{\mathrm{d}\gamma}{\mathrm{d}z}\right\} &= 0\,,
\end{aligned}$$

for the case of equilibrium, and for the case of motion such as that just considered it will only be necessary to replace A by mA in these equations. If we measure the angles of which l, m, n are the cosines from the external normal, the forces X_1, Y_1, Z_1 must be reckoned positive when l, m, and n being positive, the surface of the solid is urged in the negative directions of x, y, z, and in other cases the signs must be taken conformably.

If the solid considered is in a state of constraint when at rest, and is moreover put into a state of vibration, the pressures and displacements due to these two causes must be calculated separately and added together. If m were equal to 1, they could be calculated together from the same equations.

5. Principles of Poisson's theory of elastic solids, and of the oblique pressures existing in fluids in motion. Objections to one of his hypotheses. Reflections on the constitution, and equations of motion of the luminiferous ether in vacuum

17. In the twentieth *Cahier* of the *Journal de l'École Polytechnique* may be found a memoir by POISSON, entitled *Mémoire sur les Equations générales de l'Equilibre et du Mouvement des Corps solides élastiques et des Fluides*[1], which contains the substance of two memoirs presented by him to the Academy, brought together with some additions. In this memoir the author treats principally of the equations of equilibrium and motion of elastic solids, of the equations of equilibrium of fluids, with reference especially to capillary attraction, and of the equations of motion of fluids supposing the pressure not to be equal in all directions.

It is supposed by POISSON that all bodies, whether solid or fluid, are composed of ultimate molecules, separated from each other by vacant spaces. In the cases of an uncrystallized solid in its natural state, and of a fluid in equilibrium, he supposes that the molecules are arranged irregularly, and that the average arrangement is the same in all directions. These molecules he supposes to act on each other with forces, of which the main part is a force in the direction of the line joining the centres of gravity, and varying as some function of the distance of these points, and the remainder a secondary force, or it may be two secondary forces, depending on the molecules not being mathematical points. He supposes that it is on these secondary forces that the solidity of solid bodies depends. He supposes however that in calculating the pressures these secondary forces may be neglected, partly because they become insensible at much smaller distances than the main part of

[1] 1831, `https://gallica.bnf.fr/ark:/12148/bpt6k433676w`

the forces, and partly because they act, on the average, alike in all directions. He supposes that the molecular force decreases very rapidly as the distance increases, yet not so rapidly but that the sphere in which the molecular action is sensible contains an immense number of molecules. He supposes consequently that in estimating the resultant force of a hemisphere of the medium on a molecule in the centre of its base the action of the neighbouring molecules, which are situated irregularly, may be neglected compared with the action of those more remote, of which the average may be taken. The consequence of this supposition of course is that the total action is normal to the base of the hemisphere, and sensibly the same for one molecule as for an adjacent one.
The rest of the reasoning by which POISSON establishes the equations of motion and equilibrium of elastic solids is purely mathematical, sufficient data having been already assumed. It might appear that the reasoning in Art. 16 of his memoir, by which the expression for N is simplified, required the fresh hypothesis of a symmetrical arrangement of the molecules; but it really does not, being admissible according to the principle of averages. Taking for the natural state of the body that in which the pressure is zero, the equations at which POISSON arrives contain only *one* unknown constant k, whereas the equations of chapter 4 of this paper contain *two*, A or mA and B. This difference depends on the assumption made by POISSON that the irregular part of the force exerted by a hemisphere of the medium on a molecule in the centre of its base may be neglected in comparison with the whole force. As a result of this hypothesis, POISSON finds that the change in direction, and the proportionate change in length, of a line joining two molecules are continuous functions of the co-ordinates of one of the molecules and the angles which determine the direction of the line; whereas in chapter 4, if we adopt the hypothesis of ultimate molecules at all, it is allowable to suppose that these quantities vary irregularly in passing from one pair of molecules to an adjacent pair. Of course the equations of chapter 4 ought to reduce themselves to POISSON's equations for a particular relation between A and B. Neglecting the heat developed by compression, as POISSON has done, and therefore putting $m = 1$, this relation is $A = 5B$.

18. POISSON's theory of fluid motion is as follows. The time t is supposed to be divided into a number n of equal parts, each equal to τ. In the first of these the fluid is supposed to be displaced as an elastic solid would be, according to POISSON's previous theory, and therefore the pressures are given by the same equations. If the causes producing the displacement were now to cease, the fluid would re-arrange itself, so that the average arrangement about each point should be the same in all directions after a very short time. During this time, the pressures would have altered, in an unknown manner, from those corresponding to a displaced solid to a normal pressure equal to $p + \frac{\mathrm{D}p}{\mathrm{D}t} \cdot \tau$, the pressures during the alteration involving an unknown function of the time elapsed since the end of the interval τ. Another displacement and another re-arrangement may now be supposed to take place, and

so on. But since these very small relative motions will take place independently of each other, we may suppose each displacement to begin at the expiration of the time during which the preceding one is supposed to remain, and we may suppose each re-arrangement to be going on during the succeeding displacements. Supposing now n to become infinite, we pass to the case in which the fluid is supposed to be continually beginning to be displaced as a solid would, and continually re-arranging itself so as to make the average arrangement about each point the same in all directions.

Poisson's equations (9), page 152, which are applicable to the motion of a liquid, or of an elastic fluid, in which the change of density is small, agree with Equation 2.12 of this paper. For the quantity ψt is the pressure p which would exist at any instant if the motion were then to cease, and the increment, $\frac{\mathrm{d}\psi t}{\mathrm{d}t} \cdot \tau$ or $\frac{\mathrm{D}p}{\mathrm{D}t} \cdot \tau$, of this quantity in the very small time τ will depend only on the increment, $\frac{\mathrm{d}\chi t}{\mathrm{d}t} \cdot \tau$ or $\frac{\mathrm{D}\rho}{\mathrm{D}t} \cdot \tau$, of the density χt or ρ. Consequently the value of $\frac{\mathrm{d}\psi t}{\mathrm{d}t} \cdot \tau$ will be the same as if the density of the particle considered passed from χt to $\chi t + \frac{\mathrm{d}\chi t}{\mathrm{d}t} \cdot \tau$ in the time τ by a uniform motion of dilatation. I suppose that according to Poisson's views such a motion would not require a re-arrangement of the molecules, since the pressure remains equal in all directions. On this supposition we shall get the value of $\frac{\mathrm{d}\psi t}{\mathrm{d}t} \cdot \tau$ from that of $R'_\prime - K$ in the equations of page 140 by putting

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y} = \frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{1}{3\chi t} \cdot \frac{\mathrm{d}\chi t}{\mathrm{d}t} \,.$$

We have therefore

$$\alpha \cdot \frac{\mathrm{d}\psi t}{\mathrm{d}t} = \frac{\alpha}{3} \cdot (K - 5k) \cdot \frac{1}{\chi t} \frac{\mathrm{d}\chi t}{\mathrm{d}t} \,.$$

Putting now for $\beta + \beta'$ its value $2\alpha k$, and for $\frac{1}{\chi t} \frac{\mathrm{d}\chi t}{\mathrm{d}t}$ its value given by equation (2), the expression for ϖ, page 152, becomes

$$\varpi = p + \frac{\alpha}{3} \cdot (K + k) \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z} \right) \,.$$

Observing that $\alpha \cdot (K + k) = \beta$, this value of ϖ reduces Poisson's equations (9) to the Equation 2.12 of this paper.

Poisson himself has not made this reduction of his equations, nor any equivalent one, so that his equations, as he has left them, involve two arbitrary constants. The reduction of these two to one depends on the assumption that a uniform expansion of any particle does not require a re-arrangement of the molecules, as it leaves the pressure still equal in all directions. If we do not make this assumption, but retain the two arbitrary constants, the equations will be the same as those which would be obtained by the method of this paper, supposing the quantity κ of Art. 3, page 82 not to be zero.

19. There is one hypothesis made in the common theory of elastic solids, the truth of which appears to me very questionable. That hypothesis is the one to which I have already alluded in Art. 17, page 113, respecting the legitimacy of neglecting the irregular part of the action of the molecules in the immediate neighbourhood of the one considered, in comparison with the total action of those more remote, which is regular. It is from this hypothesis that it follows as a result that the molecules are not displaced among one another in an irregular manner, in consequence of the directive action of neighbouring molecules. Now it is obvious that the molecules of a fluid admit of being displaced among one another with great readiness. The molecules of solids, or of most solids at any rate, must admit of new arrangements, for most solids admit of being bent, permanently, without being broken. Are we then to suppose that when a solid is constrained it has no *tendency* to relieve itself from the state of constraint, in consequence of its molecules *tending towards* new relative positions, provided the amount of constraint be very small? It appears to me to be much more natural to suppose *à priori* that there should be some such tendency.

In the case of a uniform dilatation or contraction of a particle, a re-arrangement of its molecules would be of little or no avail towards relieving it from constraint, and therefore it is natural to suppose that in this case there is little or no tendency towards such a re-arrangement. It is quite otherwise, however, in the case of what I have called a displacement of shifting. Consequently B will be less than if there were no tendency to a re-arrangement. On the hypothesis mentioned in this article, of which the absence of such tendency is a consequence, I have said that a relation has been found between A and B, namely $A = 5B$. It is natural then to expect to find the ratio of A to B greater than 5, approaching more nearly to 5 as the solid considered is more hard and brittle, but differing materially from 5 for the softer solids, especially such as India rubber, or, to take an extreme case, jelly. According to this view the relation $A = 5B$ belongs only to an ideal elastic solid, of which the solidity, or whatever we please to call the property considered, is absolutely perfect.

To shew how implicitly the common theory of elasticity seems to be received by some, I may mention that MM. Lamé and Clapeyron mention India rubber among the substances to which it would seem they consider their theory applicable[2]. I do not know whether the coefficient of elasticity, according to that theory, has been determined experimentally for India rubber, but one would fancy that the cubical compressibility thence deduced, by a method which will be seen in the next article, would turn out comparable with that of a gas.

20. I am not going to enter into the solution of Equation 4.2, but I wish to make a few remarks on the results in some simple cases. If k be the cubical contraction

[2] *Mémoires presentés à L'Institut*, Tom. IV. p. 469., `https://www.biodiversitylibrary.org/page/27262845`

due to a uniform pressure P, then will

$$k = \frac{3P}{A}\,.$$

If a wire or rod, of which the boundary is any cylindrical surface, be pulled in the direction of its length by a force of which the value, referred to a unit of surface of a section of the rod, in P, the rod will extend itself uniformly in the direction of its length, and contract uniformly in the perpendicular direction; and if e be the extension in the direction of the length, and c the contraction in any perpendicular direction, both referred to a unit of length, we shall have

$$e = \frac{A+B}{3AB}\cdot P\,,$$
$$c = \frac{A-2B}{6AB}\cdot P\ :$$

also, the cubical dilatation is

$$e - 2c = \frac{P}{A}\,.$$

If a cylindrical wire of radius r be twisted by a couple of which the moment is M, and if θ be the angle of torsion for a length z of the wire, we shall have

$$\theta = \frac{2Mz}{\pi B r^4}\,.$$

The expressions for k, c, e and θ, and of course all expressions of the same nature, depend on the reciprocals of A and B. Suppose now the value of e, or θ, or any similar quantity not depending on A alone, be given as the result of observation. It will easily be conceived that we might find very nearly the same value for B whether we supposed $A = 5B$ or $A = nB$, where n may be considerably greater than 5, or even infinite. Consequently the observation of two such quantities, giving very nearly the same value of B, might be regarded as confirming the common equations.

If we denote by E the coefficient of elasticity when A is supposed to be equal to $5B$ we have, neglecting the atmospheric pressure[3]

$$e = \frac{2P}{5E}$$
$$\theta = \frac{2Mz}{\pi E r^4}\,.$$

If now we denote by E_1 the value of E deduced from observation of the value of e, and by E_2 the value of E obtained by observing the value of θ, or else, which

[3] Lamé, *Cours de Physique*, Tom. I.

comes to the same, by observing the time of oscillation of a known body oscillating by torsion, we shall have

$$\frac{2}{5E_1} = \frac{1}{3} \cdot \left(\frac{1}{A} + \frac{1}{B} \right) , \quad E_2 = B , \quad \text{whence } \frac{1}{A} = \frac{6}{5E_1} - \frac{1}{E_2} .$$

If A be greater than $5B$, E_1 ought to be a little greater than E_2. This appears to agree with observation. Thus the following numbers are given by M. Lamé[4] $E_1 = 8000$, $E_2 = 7500$ for iron; $E_1 = 2510$, $E_2 = 2250$ for brass[5]. The difference between the values of E_1 and E_2 is attributed by M. Lamé to the errors to which the observation of the small quantity e is liable. If the above numbers may be trusted, we shall have

$$A = 60000 , \quad B = 7500 , \quad \frac{A}{B} = 8 \text{ for iron} ;$$
$$A = 29724 , \quad B = 2250 , \quad \frac{A}{B} = 13.21 \text{ for brass} .$$

The cubical contraction k is almost too small to be made the subject of direct observation[6], it is therefore usually deduced from the value of e, or from the coefficient of elasticity E found in some other way. On the supposition of a single coefficient E, we have

$$\frac{k}{e} = \frac{3}{2}$$

but retaining the two, A and B, we have

$$\frac{k}{e} = \frac{9B}{A+B} = 9 \cdot \frac{1}{\left(1 + \frac{B}{A}\right)} \cdot \frac{B}{A} ,$$

which will differ greatly from $\frac{3}{2}$ if $\frac{A}{B}$ be much greater than 5. The whole subject therefore requires, I think, a careful examination, before we can set down the values of the coefficients of cubical contraction of different substances in the list of well ascertained physical data. The result, which is generally admitted, that the ratio of the velocity of propagation of normal, to that of tangential vibrations in a solid is equal to $\sqrt{3}$, is another which depends entirely on the supposition that $A = 5B$. The value of m, again, as deduced from observation, will depend upon the ratio of A to B; and it would be highly desirable to have an accurate list of the values

[4]Lamé, *Cours de Physique de l'École polytechnique*, Tom. I. 1836, `https://books.google.bi/books?id=0sxHAAAAIAAJ&hl=fr&source=gbs_navlinks_s`

[5]These numbers refer to the French units of length and weight.

[6]I find however that direct experiments have been made by Prof. Oersted. According to these experiments the cubical compressibility of solids which would be obtained from Poisson's theory is in some cases as much as 20 or 30 times too great. See the *Report of the British Association* for 1833, p. 353, or *Archives des découvertes*, &c. for 1834, p. 94. (It is to be noted that Oersted's method gives only *differences* of compressibility.)

of m for different substances, in hopes of thereby discovering in what manner the action of heat on those substances is related to the physical constants belonging to them, such as their densities, atomic weights, &c.
The observations usually made on elastic solids are made on slender pieces, such as wires, rods, and thin plates. In such pieces, all the particles being at no great distance from the surface, it is easy to see that when any small portion is squeezed in one direction it has considerable liberty of expanding itself in a direction perpendicular to this, and consequently the results must depend mainly on the value of B, being not very different from what they would be if A were infinite. This is not so much the case with thick, stout pieces. If therefore such pieces could be put into a state of isochronous vibration, so that the musical notes and nodal lines could be observed, they would probably be better adapted than slender pieces for determining the value of mA. The value of m might be determined by comparing the value of mA, deduced from the observation of vibrations, with the value of A, deduced from observations made in cases of equilibrium, or, perhaps, of very slow motion.

21. Equation 4.4 are the same as those which have been obtained by different authors as the equations of motion of the luminiferous ether in vacuum. Assuming for the present that the equations of motion of this medium ought to be determined on the same principles as the equations of motion of an elastic solid, it will be necessary to consider whether the Equation 4.4 are altered by introducing the consideration of a uniform pressure Π existing in the medium when in equilibrium; for we have evidently no right to assume, either that no such pressure exists, or, supposing it to exist, that the medium would expand itself but very slightly if it were removed. It will now no longer be allowable to confound the pressure referred to a unit of surface as it was, in the position of equilibrium of the medium, with the pressure referred to a unit of surface as it actually is. The latter mode of referring the pressure is more natural, and will be more convenient. Let the pressure, referred to a unit of surface as it is, be resolved into a normal pressure $\Pi + p_1$ and a tangential pressure t_1. All the reasoning of chapter 4 will apply to the small forces p_1 and t_1; only it must be remembered that in estimating the whole oblique pressure a normal pressure Π must be compounded with the pressures given by Equation 4.3. In forming the equations of motion, the pressure Π will not appear, because the resultant force due to it acting on the element of the medium which is considered is zero. The Equation 4.4 will therefore be the equations of motion required.
If we had chosen to refer the pressure to a unit of surface in the original state of the surface, and had resolved the whole pressure into a pressure $\Pi + p_1$ normal to the original position of the surface, and a pressure t_1 tangential to that position, the reasoning of chapter 4 would still have applied, and we should have obtained the same expressions as in Equation 4.3 for the pressures P_1, T_1, &c., but the numerical value of A would have been different. According to this method, the pressure Π

would have appeared in the equations of motion. It is when the pressures are measured according to the method which I have adopted that it is true that the equilibrium of the medium would be unstable if either A or B were negative. I must here mention that from some oversight the right-hand sides of POISSON's equations, at page 68 of the memoir to which I have referred, are wrong. The first of these equations ought to contain $\frac{\Pi}{\rho} \cdot \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} \right)$, instead or $\frac{\Pi}{\rho} \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}$, and similar changes must be made in the other two equations.

It is sometimes brought as an objection to the equations of motion of the luminiferous ether, that they are the same as those employed for the motion of solid bodies, and that it seems unnatural to employ the same equations for substances which must be so differently constituted. It was, perhaps, in consequence of this objection that POISSON proposes, at page 147 of the memoir which I have cited, to apply to the calculation of the motion of the luminiferous ether the same principles, with a certain modification, as those which he employed in arriving at his equations (9) page 152, i.e. the Equation 2.12 of this paper. That modification consists in supposing that a certain function of the time $\phi(t)$ does not vary very rapidly compared with the variation of the pressure. Now the law of the transmission of a motion transversal to the direction of propagation depending on Equation 2.12 of this paper is expressed, in the simplest case, by the Equation 3.1; and we see that this law is the same as that of the transmission of heat, a law extremely different from that of the transmission of vibratory motions. It seems therefore unlikely that these principles are applicable to the calculation of the motion of light, unless the modification which I have mentioned be so great as wholly to alter the character of the motion, that is, unless we suppose the pressure to vary extremely fast compared with the function $\phi(t)$, whereas in ordinary cases of the motion of fluids the function $\phi(t)$ is supposed to vary extremely fast compared with the pressure.

Another view of the subject may be taken which I think deserves notice. Before explaining this view however it will be necessary to define what I mean in this chapter by the word *elasticity*. There are two distinct kinds of elasticity; one, that by which a body which is uniformly compressed tends to regain its original volume, the other, that by which a body which is constrained in a manner independent of compression tends to assume its original form. The constants A and B of chapter 4 may be taken as measures of these two kinds of elasticity. In the present chapter, the word will be used to denote the second kind. Now many highly elastic substances, as iron, copper, &c., are yet to a very sensible degree plastic. The plasticity of lead is greater than that of iron or copper, and, as appears from experiment, its elasticity less. On the whole it is probable that the greater the plasticity of a substance the less its elasticity, and *vice versa*, although this rule is probably far from being without exception. When the plasticity of the substance is still further increased, and its elasticity diminished, it passes into a viscous fluid. There seems no line of demarcation between a solid and a viscous fluid. In fact,

the practical distinction between these two classes of bodies seems to depend on the intensity of the *extraneous* force of gravity, compared with the intensity of the forces by which the parts of the substance are held together. Thus, what on the Earth is a soft solid might, if carried to the Sun, and retained at the same temperature, be a viscous fluid, the force of gravity at the surface of the Sun being sufficient to make the substance spread out and become level at the top: while what on the Earth is a viscous fluid might on the surface of Pallas be a soft solid. The gradation of viscous, into what are called perfect fluids seems to present as little abruptness as that of solids into viscous fluids; and some experiments which have been made on the sudden conversion of water and ether into vapour, when enclosed in strong vessels and exposed to high temperatures, go towards breaking down the distinction between liquids and gases.

According to the law of continuity, then, we should expect the property of elasticity to run through the whole series, only, it may become insensible, or else may be masked by some other more conspicuous property. It must be remembered that the elasticity here spoken of is that which consists in the tangential force called into action by a displacement of continuous sliding: the displacements also which will be spoken of in this chapter must be understood of such displacements as are independent of condensation or rarefaction. Now the distinguishing property of fluids is the extreme mobility of their parts. According to the views explained in this article, this mobility is merely an extremely great plasticity, so that a fluid admits of a *finite*, but exceedingly small amount of constraint before it will be relieved from its state of tension by its molecules assuming new positions of equilibrium. Consequently the same oblique pressures can be called into action in a fluid as in a solid, provided the amount of relative displacement of the parts be exceedingly small. All we know for certain is that the effect of elasticity in fluids, (elasticity of the second kind be it remembered,) is quite insensible in cases of equilibrium, and it is probably insensible in all ordinary cases of fluid motion. Should it be otherwise, Equation 2.8 and Equation 2.12 will not be true, or only approximately true. But a little consideration will shew that the property of elasticity may be quite insensible in ordinary cases of fluid motion, and may yet be that on which the phenomena of light entirely depend. When we find a vibrating string, the small extent of vibration of which can be actually seen, filling a whole room with sound, and remember how rapidly the intensity of the vibrations of the air must diminish as the distance from the string increases[7], we may easily conceive how small in general must be the amount of the relative motion of adjacent particles of air in the case of sound. Now the extent of the vibration of the ether, in the case of light, may be as small compared with the length of a wave of light as that of the air is compared with the length of a wave of sound: we have no reason to suppose

[7]In all ordinary cases it is to the vibrations of the sounding-board, or of the supporting body acting as a sounding-board, and not to those of the string directly, that the sound is almost wholly due.

it otherwise. When we remember then that the length of a wave of sound in air varies from some inches to several feet, while the greatest length of a wave of light is about 0.00003[8] of an inch, it is easy to imagine that the *relative* displacement of the particles of ether may be so small as not to reach, nor even come near to the greatest relative displacement which could exist without the molecules of the medium assuming new positions of equilibrium, or, to keep clear of the idea of molecules, without the medium assuming a new arrangement which might be permanent.

It has been supposed by some that air, like the luminiferous ether, ought to admit of transversal vibrations. According to the views of this article such would, mathematically speaking, be the case; but the extent of such vibrations would be necessarily so very small as to render them utterly insensible, unless we had organs with a delicacy equal to that of the retina adapted to receive them.

It has been shewn to be highly probable that the ratio of A to B increases rapidly according as the medium considered is softer and more plastic. For fluids therefore, and among them for the luminiferous ether, we should expect the ratio of A to B to be extremely great. Now if N be the velocity of propagation of normal vibrations in the medium considered in chapter 4, and T that of transversal vibrations, it may be shewn from Equation 4.4 that

$$N^2 = \frac{mA + 4B}{3\rho}\,,$$
$$T^2 = \frac{B}{\rho}\,.$$

This is very easily shewn in the simplest case of plane waves : for if $\beta = \gamma = 0$, $\alpha = f(x)$, the Equation 4.4 give

$$\rho \cdot \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \frac{1}{3} \cdot (mA + 4B) \cdot \frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2}\,,$$

whence

$$\alpha = \phi(Nt - x) + \psi(Nt + x)\,;$$

and if $\alpha = \gamma = 0$, $\beta = f(x)$, the same equations give

$$\rho \cdot \frac{\mathrm{d}^2\beta}{\mathrm{d}t^2} = B \cdot \frac{\mathrm{d}^2\beta}{\mathrm{d}x^2}\,,$$

whence

$$\beta = \zeta(Tt - x) + \xi(Tt + x)\,.$$

Consequently we should expect to find the ratio of N to T extremely great. This

[8]Equals 0.762 µm

agrees with a conclusion of the late Mr. GREEN's[9]. Since the equilibrium of any medium would be unstable if either A or B were negative, the least possible value of the ratio of N^2 to T^2 is $\frac{4}{3}$, a result at which Mr. GREEN also arrived. As however it has been shewn to be highly probable that $A > 5B$ even for the hardest solids, while for the softer ones $\frac{A}{B}$ is much greater than 5, it is probable that $\frac{N}{T}$ is greater than $\sqrt{3}$ for the hardest solids, and much greater for the softer ones.
If we suppose that in the luminiferous ether $\frac{A}{B}$ may be considered infinite, the equations of motion admit of a simplification. For if we put

$$mA \cdot \left(\frac{\mathrm{d}\alpha}{\mathrm{d}x} + \frac{\mathrm{d}\beta}{\mathrm{d}y} + \frac{\mathrm{d}\gamma}{\mathrm{d}z} \right) = -p$$

in Equation 4.4, and suppose mA to become infinite while p remains finite, the equations become

$$\left.\begin{aligned} \rho \cdot \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} &= -\frac{\mathrm{d}p}{\mathrm{d}x} + B \cdot \left(\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2} + \frac{\mathrm{d}^2\alpha}{\mathrm{d}y^2} + \frac{\mathrm{d}^2\alpha}{\mathrm{d}z^2} \right) \text{ , \&c.} \\ \text{and } \ 0 &= \frac{\mathrm{d}\alpha}{\mathrm{d}x} + \frac{\mathrm{d}\beta}{\mathrm{d}y} + \frac{\mathrm{d}\gamma}{\mathrm{d}z} \, . \end{aligned}\right\} \tag{5.1}$$

When a vibratory motion is propagated in a medium of which Equation 5.1 are the equations of motion, it may be shewn that $p = \psi(t)$, if the medium be indefinitely extended, or else if there be no motion at its boundaries. In considering therefore the transmission of light in an uninterrupted vacuum the terms involving p will disappear from Equation 5.1; but these terms are, I believe, important in explaining Diffraction, which is the principal phenomenon the laws of which depend only on the equations of motion of the luminiferous ether in vacuum. It will be observed that putting $A = \infty$ comes to the same thing as regarding the ether as incompressible with respect to those motions which constitute Light.

[9] *Transactions of the Cambridge Philosophical Society*, Vol. VII. Part I., p. 2., 1819, https://www.biodiversitylibrary.org/page/2397826

Part III.

On the Effect of the Internal Friction of Fluids on the Motion of Pendulums

1. Introductory Notes

The great importance of the results obtained by means of the pendulum has induced philosophers to devote so much attention to the subject, and to perform the experiments with such a scrupulous regard to accuracy in every particular, that pendulum observations may justly be ranked among those most distinguished by modern exactness. It is unnecessary here to enumerate the different methods which have been employed, and the several corrections which must be made, in order to deduce from the actual observations the result which would correspond to the ideal case of a simple pendulum performing indefinitely small oscillations in vacuum. There is only one of these corrections which bears on the subject of the present paper, namely, the correction usually termed the *reduction to a vacuum*. On account of the inconvenience and expense attending experiments in a vacuum apparatus, the observations are usually made in air, and it then becomes necessary to apply a small correction, in order to reduce the observed result to what would have been observed had the pendulum been swung in a vacuum. The most obvious effect of the air consists in a diminution of the moving force, and consequent increase in the time of vibration, arising from the buoyancy of the fluid. The correction for buoyancy is easily calculated from the first principles of hydrostatics, and formed for a considerable time the only correction which it was thought necessary to make for reduction to a vacuum. But in the year 1828 Bessel, in a very important memoir in which he determined by a new method the length of the seconds' pendulum, pointed out from theoretical considerations the necessity of taking account of the inertia of the air as well as of its buoyancy. The numerical calculation of the effect of the inertia forms a problem of hydrodynamics which Bessel did not attack; but he concluded from general principles that a fluid, or at any rate a fluid of small density, has no other effect on the time of very small vibrations of a pendulum than that it diminishes its gravity and increases its moment of inertia. In the case of a body of which the dimensions are small compared with the length of the suspending wire, Bessel represented the increase of inertia by that of a mass equal to k times the mass of the fluid displaced, which must be supposed to be added to the inertia of the body itself. This factor k he determined experimentally for a sphere a little more than two inches in diameter, swung in air and in water. The result for air, obtained in a rather indirect way, was $k = 0.9459$, which value Bessel in a subsequent paper increased to 0.956. A brass sphere of the above size having been swung in water with two different lengths of wire in succession gave two values of k, differing a little from each other, and equal to only about two-thirds of the value obtained for air.

The attention of the scientific world having been called to the subject by the publication of Bessel's memoir, fresh researches both theoretical and experimental soon appeared. In order to examine the effect of the air by a more direct method than that employed by Bessel, a large vacuum apparatus was erected at the expense of the Board of Longitude, and by means of this apparatus Captain (now Colonel) Sabine determined the effect of the air on the time of vibration of a particular invariable pendulum. The results of the experiments are contained in a memoir read before the Royal Society in March 1829, and printed in the *Philosophical Transactions* for that year[1]. The mean of eight very consistent experiments gave 1.655 as the factor by which for that pendulum the old correction for buoyancy must be multiplied in order to give the whole correction on account of the air. A very remarkable fact was discovered in the course of these experiments. While the effects of air at the atmospheric pressure and under a pressure of about half an atmosphere were found to be as nearly as possible proportional to the densities, it was found that the effect of hydrogen at the atmospheric pressure was much greater, compared with the effect of air, than corresponded with its density. In fact, it appeared that the ratio of the effects of hydrogen and air on the times of vibration was about 1 to $5\frac{1}{4}$, while the ratio of the densities is only about 1 to 13. In speaking of this result Colonel Sabine remarks,

"The difference of this ratio from that shewn by experiment is greater than can well be ascribed to accidental error in the experiment, particularly as repetition produced results almost identical. May it not indicate an inherent property in the elastic fluids, analogous to that of viscidity in liquids, of resistance to the motion of bodies passing through them, independently of their density? a property, in such case, possessed by air and hydrogen gas in very different degrees; since it would appear from the experiments that the ratio of the resistance of hydrogen gas to that of air is more than double the ratio following from their densities. Should the existence of such a distinct property of resistance, varying in the different elastic fluids, be confirmed by experiments now in progress with other gases, an apparatus more suitable than the present to investigate the ratio in which it is possessed by them, could scarcely be devised: and the pendulum, in addition to its many important and useful purposes in general physics, may find an application for its very delicate, but, with due precaution, not more delicate than certain, determinations, in the domain of chemistry."

Colonel Sabine has informed me that the experiments here alluded to were interrupted by a cause which need not now be mentioned, but that as far as they went they confirmed the result of the experiments with hydrogen, and pointed

[1] `https://royalsocietypublishing.org/toc/rstl/1829/119`

out the existence of a specific action in different gases, quite distinct from mere variations of density.

Our knowledge on the subject of the effect of air on the time of vibration of pendulums has received a most valuable addition from the labours of the late Mr Baily, who erected a vacuum apparatus at his own house, with which he performed many hundreds of careful experiments on a great variety of pendulums. The experiments are described in a paper read before the Royal Society on the 31st of May 1832[2]. The result for each pendulum is expressed by the value of n, the factor by which the old correction for buoyancy must be multiplied in order to give the whole effect of the air as deduced from observation. Four spheres, not quite $1\frac{1}{2}$ inch in diameter, gave as a mean $n = 1.864$, while three spheres, a little more than 2 inches in diameter, gave only 1.748. The latter were nearly of the same size as those with which Bessel, by a different method, had obtained $k = 0.946$ or 0.956, which corresponds to $n = 1.946$ or 1.956. Among the "Additional Experiments" in the latter part of Baily's paper, is a set in which the pendulums consisted of plain cylindrical rods. With these pendulums it was found that n regularly increased, though according to an unknown law, as the diameter of the rod decreased. While a brass tube $1\frac{1}{2}$ inch in diameter gave n equal to about 2.3, a thin rod or thick wire only 0.072 inch in diameter gave for n a value as great as 7.530.

Mathematicians in the meanwhile were not idle, and several memoirs appeared about this time, of which the object was to determine from hydrodynamics the effect of a fluid on the motion of a pendulum. The first of these came from the pen of the celebrated Poisson. It was read before the French Academy on the 22nd of August 1831, and is printed in the 11th Volume of the Memoirs[3]. In this paper, Poisson considers the case of a sphere suspended by a fine wire, and oscillating in the air, or in any gas. He employs the ordinary equations of motion of an elastic fluid, simplified by neglecting the terms which involve the square of the velocity; but in the end, in adapting his solution to practice, he neglects, as insensible, the terms by which alone the action of an elastic differs from that of an incompressible fluid, so that the result thus simplified is equally applicable to fluids of both classes. He finds that when insensible quantities are neglected $n = 1.5$, so that the mass which we must suppose added to that of the pendulum is equal to half the mass of the fluid displaced. This result does not greatly differ from the results obtained experimentally by Bessel in the case of spheres oscillating in water, but differs materially from the result he had obtained for air. It agrees pretty closely with some experiments which had been performed about fifty years before by Dubuat, who had in fact anticipated Bessel in shewing that the time of vibration of a pendulum vibrating in a fluid would be affected by the inertia of the fluid as well as by its density. Dubuat's labours on this subject had been altogether overlooked by

[2] https://royalsocietypublishing.org/doi/10.1098/rstl.1832.0020

[3] https://books.google.de/books?id=MV8bHLRpxmgC&hl=de&source=gbs_navlinks_s, p. 521

those who were engaged in pendulum experiments; probably because such persons were not likely to seek in a treatise on hydraulics for information connected with the subject of their researches. DUBUAT had, in fact, rather applied the pendulum to hydrodynamics than hydrodynamics to the pendulum.

In the *Philosophical Magazine* for September 1833, p. 185, is a short paper by Professor CHALLIS, on the subject of the resistance to a ball pendulum[4]. After referring to a former paper, in which he had shewn that no sensible error would be committed in a problem of this nature by neglecting the compressibility of the fluid even if it be elastic, Professor CHALLIS, adopting a particular hypothesis respecting the motion, obtains 2 for the value of the factor n for such a pendulum. This mode of solution, which is adopted in several subsequent papers, has given rise to a controversy between Professor CHALLIS and the Astronomer Royal, who maintains the justice of POISSON's result.

In a paper read before the Royal Society of Edinburgh on the 16th of December 1833, and printed in the 13th Volume of the Society's *Transactions*, GREEN has determined from the common equations of fluid motion the resistance to an ellipsoid performing small oscillations without rotation[5]. The result is expressed by a definite integral; but when two of the principal axes of the ellipsoid become equal, the integral admits of expression in finite terms, by means of circular or logarithmic functions. When the ellipsoid becomes a sphere, GREEN's result reduces itself to POISSON's.

In a memoir read before the Royal Academy of Turin on the 18th of January 1835, and printed in the 37th Volume of the memoirs of the Academy, M. PLANA has entered at great length into the theory of the resistance of fluids to pendulums[6]. This memoir contains, however, rather a detailed examination of various points connected with the theory, than the determination of the resistance for any new form of pendulum. The author first treats the case of an incompressible fluid, and then shews that the result would be sensibly the same in the case of an elastic fluid. In the case of a ball pendulum, the only one in which a complete solution of the problem is effected, M. PLANA's result agrees with POISSON's.

In a paper read before the Cambridge Philosophical Society on the 29th of May 1843, and printed in the 8th Volume of the *Transactions*, p. 105 (see Part I), I have determined the resistance to a ball pendulum oscillating within a concentric spherical envelope, and have pointed out the source of an error into which POISSON had fallen, in concluding that such an envelope would have no effect. When the radius of the envelope becomes infinite, the solution agrees with that which POISSON had obtained for the case of an unlimited mass of fluid. I have also investigated the increase of resistance due to the confinement of the fluid by a distant rigid plane. The same paper contains likewise the calculation of the resistance to a long

[4] https://www.tandfonline.com/doi/abs/10.1080/14786443308648154

[5] https://www.biodiversitylibrary.org/bibliography/29119

[6] Mémoire sur le mouvement d'un Pendule dans un milieu résistant. M. Plana, Astron. Roy. Turin, 1835

cylinder oscillating in a mass of fluid either unlimited, or confined by a cylindrical envelope, having the same axis as the cylinder in its position of equilibrium. In the case of an unconfined mass of fluid, it appeared that the effect of inertia was the same as if a mass equal to that of the fluid displaced were distributed along the axis of the cylinder, so that $n = 2$ in the case of a pendulum consisting of a long cylindrical rod. This nearly agrees with BAILY's result for the long $1\frac{1}{2}$ inch tube; but, on comparing it with the results obtained with the cylindrical rods, we observe the same sort of discrepancy between theory and observation as was noticed in the case of spheres. The discrepancy is, however, far more striking in the present case, as might naturally have been expected, after what had been observed with spheres, on account of the far smaller diameter of the solids employed.

A few years ago Professor THOMSON communicated to me a very beautiful and powerful method which he had applied to the theory of electricity, which depended on the consideration of what he called *electrical images*. The same method, I found, applied, with a certain modification, to some interesting problems relating to ball pendulums. It enabled me to calculate the resistance to a sphere oscillating in presence of a fixed sphere, or within a spherical envelope, or the resistance to a pair of spheres either in contact, or connected by a narrow rod, the direction of oscillation being, in all these cases, that of the line joining the centres of the spheres. The effect of a rigid plane perpendicular to the direction of motion is of course included as a particular case. The method even applies, as Professor THOMSON pointed out to me, to the uncouth solid bounded by the exterior segments of two intersecting spheres, provided the exterior angle of intersection be a submultiple of two right angles. A set of corresponding problems, in which the spheres are replaced by long cylinders, may be solved in a similar manner. These results were mentioned at the meeting of the British Association at Oxford in 1847, and are noticed in the volume of reports for that year, but they have not yet been published in detail.

The preceding are all the investigations that have fallen under my notice, of which the object was to calculate from hydrodynamics the resistance to a body of given form oscillating as a pendulum. They all proceed on the ordinary equations of the motion of fluids. They all fail to account for one leading feature of the experimental results, namely, the increase of the factor n with a decrease in the dimensions of the body. They recognize no distinction between the action of different fluids, except what arises from their difference of density.

In a conversation with Dr ROBINSON about seven or eight years ago on the subject of the application of theory to pendulums, he noticed the discrepancy which existed between the results of theory and experiment relating to a ball pendulum, and expressed to me his conviction that the discrepancy in question arose from the adoption of the ordinary theory of fluid motion, in which the pressure is supposed to be equal in all directions. He also described to me a remarkable experiment of Sir JAMES SOUTH's which he had witnessed. This experiment has not been published, but Sir JAMES SOUTH has kindly allowed me to mention it. When a

pendulum is in motion, one would naturally have supposed that the air near the moving body glided past the surface, or the surface past it, which comes to the same thing if the relative motion only be considered, with a velocity comparable with the absolute velocity of the surface itself. But on attaching a piece of gold leaf to the bottom of a pendulum, so as to stick out in a direction perpendicular to the surface, and then setting the pendulum in motion, Sir James South found that the gold leaf retained its perpendicular position just as if the pendulum had been at rest; and it was not till the gold leaf carried by the pendulum had been removed to some distance from the surface, that it began to lag behind. This experiment shews clearly the existence of a tangential action between the pendulum and the air, and between one layer of air and another. The existence of a similar action in water is clearly exhibited in some experiments of Coulomb's which will be mentioned in the second part of this paper, and indeed might be concluded from several very ordinary phenomena. Moreover Dubuat, in discussing the results of his experiments on the oscillations of spheres in water, notices a slight increase in the effect of the water corresponding to an increase in the time of vibration, and expressly attributes it to the *viscosity* of the fluid.

Having afterwards occupied myself with the theory of the friction of fluids, and arrived at general equations of motion, the same in essential points as those which had been previously obtained in a totally different manner by others, of which, however, I was not at the time aware, I was desirous of applying, if possible, these equations to the calculation of the motion of some kind of pendulum. The difficulty of the problem is of course materially increased by the introduction of internal friction, but as I felt great confidence in the essential parts of the theory, I thought that labour would not be ill-bestowed on the subject. I first tried a long cylinder, because the solution of the problem appeared likely to be simpler than in the case of a sphere. But after having proceeded a good way towards the result, I was stopped by a difficulty relating to the determination of the arbitrary constants, which appeared as the coefficients of certain infinite series by which the integral of a certain differential equation was expressed. Having failed in the case of a cylinder, I tried a sphere, and presently found that the corresponding differential equation admitted of integration in finite terms, so that the solution of the problem could be completely effected. The result, I found, agreed very well with Baily's experiments, when the numerical value of a certain constant was properly assumed; but the subject was laid aside for some time. Having afterwards attacked a definite integral to which Mr Airy had been led in considering the theory of the illumination in the neighbourhood of a caustic, I found that the method which I had employed in the case of this integral would apply to the problem of the resistance to a cylinder, and it enabled me to get over the difficulty with which I had before been baffled. I immediately completed the numerical calculation, so far as was requisite to compare the formulæ with Baily's experiments on cylindrical rods, and found a remarkably close agreement between theory and observation. These results were mentioned at the meeting of the British Association at Swansea in 1848, and are briefly described

in the volume of reports for that year.

The present paper is chiefly devoted to the solution of the problem in the two cases of a sphere and of a long cylinder, and to a comparison of the results with the experiments of Baily and others. Expressions are deduced for the effect of a fluid both on the time and on the arc of vibration of a pendulum consisting either of a sphere, or of a cylindrical rod, or of a combination of a sphere and a rod. These expressions contain only one disposable constant, which has a very simple physical meaning, and which I propose to call the *index of friction* of the fluid. This constant we may conceive determined by one observation, giving the effect of the fluid either on the time or on the arc of vibration of any one pendulum of one of the above forms, and then the theory ought to predict the effect both on the time and on the arc of vibration of all such pendulums. The agreement of theory with the experiments of Baily on the time of vibration is remarkably close. Even the rate of decrease of the arc of vibration, which it formed no part of Baily's object to observe, except so far as was necessary for making the small correction for reduction to indefinitely small vibrations, agrees with the result calculated from theory as nearly as could reasonably be expected under the circumstances.

It follows from theory that with a given sphere or cylindrical rod the factor n increases with the time of vibration. This accounts in a good measure for the circumstance that Bessel obtained so large a value of k for air, as is shewn at length in the present paper; though it unquestionably arose in a great degree from the increase of resistance due to the close proximity of a rigid plane to the swinging ball.

I have deduced the value of the index of friction of water from some experiments of Coulomb's on the decrement of the arc of oscillation of disks, oscillating in water in their own plane by the torsion of a wire. When the numerical value thus obtained is substituted in the expression for the time of vibration of a sphere, the result agrees almost exactly with Bessel's experiments with a sphere swung in water.

The present paper contains one or two applications of the theory of internal friction to problems which are of some interest, but which do not relate to pendulums. The resistance to a sphere moving uniformly in a fluid may be obtained as a limiting case of the resistance to a ball pendulum, provided the circumstances be such that the square of the velocity may be neglected. The resistance thus determined proves to be proportional, for a given fluid and a given velocity, not to the surface, but to the radius of the sphere; and therefore the accelerating force of the resistance increases much more rapidly, as the radius of the sphere decreases, than if the resistance varied as the surface, as would follow from the common theory. Accordingly, the resistance to a minute globule of water falling through the air with its terminal velocity depends almost wholly on the internal friction of air. Since the index of friction of air is known from pendulum experiments, we may easily calculate the terminal velocity of a globule of given size, neglecting the part of the resistance which depends upon the square of the velocity. The terminal velocity thus obtained

is so small in the case of small globules such as those of which we may conceive a cloud to be composed, that the apparent suspension of the clouds does not seem to present any difficulty. Had the resistance been determined from the common theory, it would have been necessary to suppose the globules much more minute, in order to account in this way for the phenomenon. Since in the case of minute globules falling with their terminal velocity the part of the resistance depending upon the square of the velocity, as determined by the common theory, is quite insignificant compared with the part which depends on the internal friction of the air, it follows that were the pressure equal in all directions in air in the state of motion, the quantity of water which would remain suspended in the state of cloud would be enormously diminished. The pendulum thus, in addition to its other uses, affords us some interesting information relating to the department of meteorology. The fifth section of the first part of the present paper contains an investigation of the effect of the internal friction of water in causing a series of oscillatory waves to subside. It appears from the result that in the case of the long swells of the ocean the effect of friction is insignificant, while in the case of the ripples raised by the wind on a small pool, the motion subsides very rapidly when the disturbing force ceases to act.

2. Analytical Investigations

2.1. Adaptation of the general equations to the case of the fluid surrounding a body which oscillates as a pendulum. General laws which follow from the form of the equations. Solution of the equations in the case of an oscillating plane

1. In a paper "*On the Theories of the Internal Friction of Fluids in Motion*, &c.," which the Society did me the honour to publish in the 8th Volume of their *Transactions*, I have arrived at the following equations, (Equation 2.12), for calculating the motion of a fluid when the internal friction of the fluid itself is taken into account, and consequently the pressure not supposed equal in all directions:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \rho \cdot \left(X - \frac{\mathrm{d}u}{\mathrm{d}t} - u\frac{\mathrm{d}u}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}y} + w\frac{\mathrm{d}u}{\mathrm{d}z} \right) + \mu \cdot \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} \right) + \frac{\mu}{3} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z} \right), \tag{2.1}$$

with two more equations which may be written down from symmetry:

$$\frac{\mathrm{d}p}{\mathrm{d}y} = \rho \cdot \left(Y - \frac{\mathrm{d}v}{\mathrm{d}t} - u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}v}{\mathrm{d}y} + w\frac{\mathrm{d}v}{\mathrm{d}z} \right) + \mu \cdot \left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 v}{\mathrm{d}y^2} + \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} \right) + \frac{\mu}{3} \cdot \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z} \right),$$

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \rho \cdot \left(Z - \frac{\mathrm{d}w}{\mathrm{d}t} - u\frac{\mathrm{d}w}{\mathrm{d}x} + v\frac{\mathrm{d}w}{\mathrm{d}y} + w\frac{\mathrm{d}w}{\mathrm{d}z} \right) + \mu \cdot \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 w}{\mathrm{d}y^2} + \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} \right) + \frac{\mu}{3} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z} \right),$$

In these equations u, v, w are the components of the velocity along the rectangular axes of x, y, z; X, Y, Z are the components of the accelerating force; p is the pressure, t the time, ρ the density, and μ a certain constant depending on the nature of the fluid.

The three equations of which Equation 2.1 is the type are not the general equations of motion which apply to a heterogeneous fluid when internal friction is taken into account, which are Equation 2.10 in my former paper, but are applicable to a homogeneous incompressible fluid, or to a homogeneous elastic fluid subject to small variations of density, such as those which accompany sonorous vibrations. It must be understood to be included in the term *homogeneous* that the temperature is uniform throughout the mass, except so far as it may be raised or lowered by sudden condensation or rarefaction in the case of an elastic fluid. The general equations contain the differential coefficients of the quantity μ with respect to x, y, and z; but the equations of the form of Equation 2.1 are in their present shape even more general than is required for the purposes of the present paper.
These equations agree in the main with those which had been previously obtained, on different principles, by Navier, by Poisson, and by M. de Saint-Venant, as I have elsewhere observed[1]. The differences depend only on the coefficient of the last term, and this term vanishes in the case of an incompressible fluid, to which Navier had confined his investigations.
The equations such as Equation 2.1 in their present shape are rather complicated, but in applying them to the case of a pendulum they may be a good deal simplified without the neglect of any quantities which it would be important to retain. In the first place the motion is supposed very small, on which account it will be allowable to neglect the terms which involve the square of the velocity. In the second place, the nature of the motion that we have got to deal with is such that the compressibility of the fluid has very little influence on the result, so that we may treat the fluid as incompressible, and consequently omit the last terms in the equations. Lastly, the forces X, Y, Z are in the present case the components of the force of gravity, and if we write

$$p + \Pi + \rho \cdot \int (X\,\mathrm{d}x + Y\,\mathrm{d}y + Z\,\mathrm{d}z)$$

for p, we may omit the terms X, Y, Z.
If z' be measured vertically downwards from a horizontal plane drawn in the neighbourhood of the pendulum, and if g be the force of gravity, $\int (X\,\mathrm{d}x + Y\,\mathrm{d}y + Z\,\mathrm{d}z) = gz'$, the arbitrary constant, or arbitrary function of the time if it should be found necessary to suppose it to be such, being included in Π. The part of the whole force acting on the pendulum which depends on the terms $\Pi + g\rho z'$ is simply a force equal to the weight of the fluid displaced, and acting vertically upwards through the centre of gravity of the volume.
When simplified in the manner just explained, the equations such as Equation 2.1

[1] Report on recent researches in Hydrodynamics. Report of the British Association for 1846, p. 16.

become

$$\left.\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}x} &= \mu \cdot \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2u}{\mathrm{d}y^2} + \frac{\mathrm{d}^2u}{\mathrm{d}z^2}\right) - \rho \cdot \frac{\mathrm{d}u}{\mathrm{d}t}\,, \\
\frac{\mathrm{d}p}{\mathrm{d}y} &= \mu \cdot \left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \frac{\mathrm{d}^2v}{\mathrm{d}y^2} + \frac{\mathrm{d}^2v}{\mathrm{d}z^2}\right) - \rho \cdot \frac{\mathrm{d}v}{\mathrm{d}t}\,, \\
\frac{\mathrm{d}p}{\mathrm{d}z} &= \mu \cdot \left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + \frac{\mathrm{d}^2w}{\mathrm{d}y^2} + \frac{\mathrm{d}^2w}{\mathrm{d}z^2}\right) - \rho \cdot \frac{\mathrm{d}w}{\mathrm{d}t}\,,
\end{aligned}\right\} \tag{2.2}$$

which, with the equation of continuity,

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z} = 0\,, \tag{2.3}$$

are the only equations which have to be satisfied at all points of the fluid, and at all instants of time.
In applying Equation 2.2 to a particular pendulum experiment, we may suppose μ constant; but in order to compare experiments made in summer with experiments made in winter, or experiments made under a high barometer with experiments made under a low, it will be requisite to regard μ as a quantity which may vary with the temperature and pressure of the fluid. As far as the result of a single experiment[2], which has been already mentioned, performed with a single elastic fluid, namely air, justifies us in drawing such a general conclusion, we may assert that for a given fluid at a given temperature μ varies as ρ.

2. For the formation of the equations such as Equation 2.1, I must refer to my former paper; but it will be possible, in a few words, to enable the reader to form a clear idea of the meaning of the constant μ.
Conceive the fluid to move in planes parallel to the plane of xy, the motion taking place in a direction parallel to the axis of y. The motion will evidently consist of a sort of continuous sliding, and the differential coefficient $\frac{\mathrm{d}v}{\mathrm{d}z}$ may be taken as a measure of the rate of sliding. In the theory it is supposed that in general the pressure about a given point is compounded of a normal pressure, corresponding to the density, which being normal is necessarily equal in all directions, and of an oblique pressure or tension, altering from one direction to another, which is expressed by means of linear functions of the nine differential coefficients of the first order of u, v, w with respect to x, y, z, which define the state of relative motion at any point of the fluid. Now in the special case considered above, if we confine our attention to one direction, that of the plane of xy, the total pressure referred to a unit of surface is compounded of a normal pressure corresponding to the density,

[2]The first of the experiments described in Col. SABINE's paper, in which the gauge stood as high as 7 inches, leads to the same conclusion; but as the vacuum apparatus had not yet been made stanch it is perhaps hardly safe to trust this experiment in a question of such delicacy.

and a tangential pressure expressed by $\mu \cdot \frac{\mathrm{d}v}{\mathrm{d}z}$, which tends to reduce the relative motion.
In the solution of Equation 2.2, μ always appears divided by ρ. Let $\mu = \mu' \cdot \rho$. The constant μ' may conveniently be called the *index of friction* of the fluid, whether liquid or gas, to which it relates. As regards its dimensions, it expresses a moving force divided by the product of a surface, a density, and the differential coefficient of a velocity with respect to a line. Hence μ' is the square of a line divided by a time, whence it will be easy to adapt the numerical value of μ' to a new unit of length or of time.

3. Besides the general Equation 2.2 and Equation 2.3, it will be requisite to consider the equations of condition at the boundaries of the fluid. For the purposes of the present paper there will be no occasion to consider the case of a free surface, but only that of the common surface of the fluid and a solid. Now, if the fluid immediately in contact with a solid could flow past it with a finite velocity, it would follow that the solid was infinitely smoother with respect to its action on the fluid than the fluid with respect to its action on itself. For, conceive the elementary layer of fluid comprised between the surface of the solid and a parallel surface at a distance h, and then regard only so much of this layer as corresponds to an elementary portion $\mathrm{d}S$ of the surface of the solid. The impressed forces acting on the fluid element must be in equilibrium with the effective forces reversed. Now conceive h to vanish compared with the linear dimensions of $\mathrm{d}S$, and lastly let $\mathrm{d}S$ vanish[3]. It is evident that the conditions of equilibrium will ultimately reduce themselves to this, that the oblique pressure which the fluid element experiences on the side of the solid must be equal and opposite to the pressure which it experiences on the side of the fluid. Now if the fluid could flow past the solid with a finite velocity, it would follow that the tangential pressure called into play by the continuous sliding of the fluid over itself was no more than counteracted by the abrupt sliding of the fluid over the solid. As this appears exceedingly improbable *a priori*, it seems reasonable in the first instance to examine the consequences of supposing that no such abrupt sliding takes place, more especially as the mathematical difficulties of the problem will thus be materially diminished. I shall assume, therefore, as the conditions to be satisfied at the boundaries of the fluid, that the velocity of a fluid particle shall be the same, both in magnitude and direction, as that of the solid particle with which it is in contact. The agreement of the results thus obtained with observation will presently appear to be highly satisfactory. When the fluid, instead of being confined within a rigid envelope, extends indefinitely around the oscillating body, we must introduce into the solution the condition that the motion

[3]To be quite precise it would be necessary to say, Conceive h and $\mathrm{d}S$ to vanish together, h vanishing compared with the linear dimensions of $\mathrm{d}S$; for so long as $\mathrm{d}S$ remains finite we cannot suppose h to vanish altogether, on account of the curvature of the elementary surface. Such extreme precision in unimportant matters tends, I think, only to perplex the reader, and prevent him from entering so readily into the spirit of an investigation.

shall vanish at an infinite distance, which takes the place of the condition to be satisfied at the surface of the envelope.
To complete the determination of the arbitrary functions which would be contained in the integrals of Equation 2.2 and Equation 2.3, it would be requisite to put $t = 0$ in the general expressions for u, v, w, obtained by integrating those equations, and equate the results to the initial velocities supposed to be given. But it would be introducing a most needless degree of complexity into the solution to take account of the initial circumstances, nor is it at all necessary to do so for the sake of comparison of theory with experiment. For in a pendulum experiment the pendulum is set swinging and then left to itself, and the first observation is not taken till several oscillations have been completed, during which any irregularities attending the initial motion would have had time to subside. It will be quite sufficient to regard the motion as already going on, and limit the calculation to the determination of the simultaneous periodic movements of the pendulum and the surrounding fluid. The arc of oscillation will go on slowly decreasing, but it will be so nearly constant for several successive oscillations that it may be regarded as strictly such in calculating the motion of the fluid; and having thus determined the resultant action of the fluid on the solid we may employ the result in calculating the decrement of the arc of oscillation, as well as in calculating the time of oscillation. Thus the assumption of periodic functions of the time in the expressions for u, v, w will take the place of the determination of certain arbitrary functions by means of the initial circumstances.

4. Imagine a plane drawn perpendicular to the axis of x through the point in the fluid whose co-ordinates are x, y, z. Let the oblique pressure in the direction of this plane be decomposed into three pressures, a normal pressure, which will be in the direction of x, and two tangential pressures in the directions of y and z respectively. Let P_1 be the normal pressure, and T_3 the tangential pressure in the direction of y, which will be equal to the component in the direction of x of the oblique pressure on a plane drawn perpendicular to the axis of y. Then by Equation 2.7 and Equation 2.8 of my former paper, and Equation 2.3 of the present,

$$P_1 = p - 2\mu \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\,, \tag{2.4}$$

$$T_3 = -\mu \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}x}\right)\,. \tag{2.5}$$

These formulæ will be required in finding the resultant force of the fluid on the pendulum, after the motion of the fluid has been determined in terms of the quantities by which the motion of the pendulum is expressed.

5. Before proceeding to the solution of the Equation 2.2 and Equation 2.3 in particular cases, it will be well to examine the general laws which follow merely

from the dimensions of the several terms which appear in the equations.
Consider any number of similar systems, composed of similar solids oscillating in a similar manner in different fluids or in the same fluid. Let a, a', a'', ... be homologous lines in the different systems; T, T', T'', ... corresponding times, such for example as the times of oscillation from rest to rest. Let x, y, z be measured from similarly situated origins, and in corresponding directions, and t from corresponding epochs, such for example as the commencements of oscillations when the systems are beginning to move from a given side of the mean position.
The form of equations Equation 2.2 and Equation 2.3 shews that the equations being satisfied for one system will be satisfied for all the systems provided

$$u \propto v \propto w\,, \quad x \propto y \propto z$$

and

$$p \propto \frac{\mu u}{x} \propto \frac{\rho u x}{t}\,.$$

The variations $x \propto y \propto z$ merely signify that we must compare similarly situated points in inferring from the circumstance that Equation 2.2 and Equation 2.3 are satisfied for one system that they will be satisfied for all the systems. If c, c', c'', ... be the maximum excursions of similarly situated points of the fluids

$$u \propto \frac{c}{T}\,, \quad x \propto a\,, \quad t \propto T\,,$$

and the sole condition to be satisfied, in addition to that of geometrical similarity, in order that the systems should be dynamically similar, becomes

$$\frac{a^2}{T} \propto \frac{\mu}{\rho} \quad \text{or} \quad \frac{a^2}{T} \propto \mu'\,. \tag{2.6}$$

This condition being satisfied, similar motions will take place in the different systems, and we shall have

$$p \propto \frac{\rho a c}{T^2}\,. \tag{2.7}$$

It follows from Equation 2.4 and Equation 2.5 and the other equations which might be written down from symmetry, that the pressures such as P_1, T_3 vary in the same manner as p, whence it appears from Equation 2.7 that the resultant or resultants of the pressures of the fluids on the solids, acting along similarly situated lines, which vary as $p \cdot a^2$, vary as $\rho \cdot a^3$ and $c \cdot T^{-2}$ conjointly. In other words, these resultants in two similar systems are to one another in a ratio compounded of the ratio of the masses of fluid displaced, and of the ratio of the maximum accelerating effective forces belonging to similarly situated points in the solids.

6. In order that two systems should be similar in which the fluids are confined by envelopes that are sufficiently narrow to influence the motion of the fluids, it is

necessary that the envelopes should be similar and similarly situated with respect to the solids oscillating within them, and that their linear dimensions should be in the same ratio as those of the oscillating bodies. In strictness, it is likewise necessary that the solids should be similarly situated with respect to the axis of rotation. If however two similar solids, such as two spheres, are attached to two fine wires, and made to perform small oscillations in two unlimited masses of fluid, and if we agree to neglect the effect of the suspending wires, and likewise the effect of the rotation of the spheres on the motion of the fluid, which last will in truth be exceedingly small, we may regard the two systems as geometrically similar, and they will be dynamically similar provided the condition stated in Equation 2.6 be satisfied. When the two fluids are of the same nature, as for example when both spheres oscillate in air, the condition of dynamical similarity reduces itself to this, that the times of oscillation shall be as the squares of the diameters of the spheres. If, with Bessel, we represent the effect of the inertia of the fluid on the time of oscillation of the sphere by supposing a mass equal to k times that of the fluid displaced added to the mass of the sphere, which increases its inertia without increasing its weight, we must expect to find k dependant on the nature of the fluid, and likewise on the diameter of the sphere. Bessel, in fact, obtained very different values of k for water and for air. Baily's experiments on spheres of different diameters, oscillating once in a second nearly, shew that the value of k increases when the diameter of the sphere decreases. Taking this for the present as the result of experiment, we are led from theory to assert that the value of k increases with the time of oscillation; in fact, k ought to be as much increased as if we had left the time of oscillation unchanged, and diminished the diameter in the ratio in which the square root of the time is increased. It may readily be shewn that the value of k obtained by Bessel's method, by means of a long and short pendulum, is greater than what belongs to the long pendulum, much more, greater than what belongs to the shorter pendulum, which oscillated once in a second nearly. The value of k given by Bessel is in fact considerably larger than that obtained by Baily, by a direct method, from a sphere of nearly the same size as those employed by Bessel, oscillating once in a second nearly.
The discussion of the experiments of Baily and Bessel belongs to chapter 3 of this paper. They are merely briefly noticed here to shew that some results of considerable importance follow readily from the general equations, even without obtaining any solution of them.

7. Before proceeding to the problems which mainly occupy this paper, it may be well to exhibit the solution of Equation 2.2 and Equation 2.3 in the extremely simple case of an oscillating plane.
Conceive a physical plane, which is regarded as infinite, to be situated in an unlimited mass of fluid, and to be performing small oscillations in the direction of a fixed line in the plane. Let a fixed plane coinciding with the moving plane be taken for the plane of yz, the axis of y being parallel to the direction of motion,

and consider only the portion of fluid which lies on the positive side of the plane of yz. In the present case, we must evidently have $u = 0$, $w = 0$; and p, v will be functions of x and t, which have to be determined. The Equation 2.3 is satisfied identically, and we get from Equation 2.2, putting $\mu = \mu' \cdot \rho$,

$$\left.\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}x} &= 0\,, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \mu' \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\,. \end{aligned}\right\} \tag{2.8}$$

The first of these equations gives $p =$ a constant, for it evidently cannot be a function of t, since the effect of the motion vanishes at an infinite distance from the plane; and if we include this constant in Π, we shall have $p = 0$. Let V be the velocity of the plane itself, and suppose

$$V = c \cdot \sin(nt)\,. \tag{2.9}$$

Putting in the second of Equation 2.8

$$v = X_1 \cdot \sin(nt) + X_2 \cdot \cos(nt)\,, \tag{2.10}$$

we get

$$\left.\begin{aligned} nX_1 &= \mu' \cdot \frac{\mathrm{d}^2 X_2}{\mathrm{d}x^2}\,, \\ nX_2 &= -\mu' \cdot \frac{\mathrm{d}^2 X_1}{\mathrm{d}x^2} = -\frac{\mu'^2}{n} \cdot \frac{\mathrm{d}^4 X_2}{\mathrm{d}x^4}\,. \end{aligned}\right\} \tag{2.11}$$

The last of these equations gives

$$\begin{aligned} X_2 = e^{-\sqrt{\frac{n}{2\mu'}}\cdot x} \cdot \left\{ A \cdot \sin\left(\sqrt{\frac{n}{2\mu'}} \cdot x\right) + B \cdot \cos\left(\sqrt{\frac{n}{2\mu'}} \cdot x\right)\right\} \\ + e^{\sqrt{\frac{n}{2\mu'}}\cdot x} \cdot \left\{ C \cdot \sin\left(\sqrt{\frac{n}{2\mu'}} \cdot x\right) + D \cdot \cos\left(\sqrt{\frac{n}{2\mu'}} \cdot x\right)\right\}\,. \end{aligned}$$

Since X_2 must not become infinite when $x = \infty$, we must have $C = 0$ and $D = 0$. Obtaining X_1, from the first of Equation 2.11, and substituting in Equation 2.10, we get

$$v = e^{-\sqrt{\frac{n}{2\mu'}}\cdot x} \cdot \left\{ -A \cdot \sin\left(nt - \sqrt{\frac{n}{2\mu'}} \cdot x\right) + B \cdot \cos\left(nt - \sqrt{\frac{n}{2\mu'}} \cdot x\right)\right\}\,.$$

Now by the equations of conditions assumed in Art. 3, page 138, we must have

$v = V$ when $x = 0$, whence

$$v = c \cdot e^{-\sqrt{\frac{n}{2\mu'}} \cdot x} \cdot \sin\left(nt - \sqrt{\frac{n}{2\mu'}} \cdot x\right) . \tag{2.12}$$

To find the normal and tangential components of the pressure of the fluid on the plane, we must substitute the above value of v in Equation 2.4 and Equation 2.5 and after differentiation put $x = 0$. P_1 and T_3 will then be the components of the pressure of the solid on the fluid, and therefore $-P_1$, $-T_3$, those of the pressure of the fluid on the solid. We get

$$\left.\begin{aligned} &P_1 = 0\,, \\ &T_3 = c\rho \cdot \sqrt{\frac{n\mu'}{2}} \cdot \{\sin(nt) + \cos(nt)\} = \rho \cdot \sqrt{\frac{n\mu'}{2}} \cdot \left\{V + \frac{1}{n} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}\right\} . \end{aligned}\right\} \tag{2.13}$$

The force expressed by the first of these terms tends to diminish the amplitude of the oscillations of the plane. The force expressed by the second has the same effect as increasing the inertia of the plane.

8. Equation 2.12 shews that a given phase of vibration is propagated from the plane into the fluid with a velocity $\sqrt{2\mu' n}$, while the amplitude of oscillation decreases in geometric progression as the distance from the plane increases in arithmetic. If we suppose the time of oscillation from rest to rest to be one second, then $n = \pi$; and if we suppose $\sqrt{\mu'} = 0.116$ inch, which, as will presently be seen, is about its value in the case of air, we get for the velocity of propagation 0.2908 inch per second nearly. If we enquire the distance from the plane at which the amplitude of oscillation is reduced to one half, we have only to put $\sqrt{\frac{n}{2\mu'}} \cdot x = \ln 2$, which gives, on the same suppositions as before respecting numerical values, $x = 0.06415$ inch nearly. For water the value of μ' is a good deal smaller than for air, and the corresponding value of x smaller likewise, since it varies *cæteris paribus* as $\sqrt{\mu'}$. Hence if a solid of revolution of large, or even moderately large, dimensions be suspended by a fine wire coinciding with the axis of revolution, and made to oscillate by the torsion of the wire, the effect of the fluid may be calculated with a very close degree of approximation by regarding each element of the surface of the solid as an element of an infinite plane oscillating with the same linear velocity.
For example, let a circular disk of radius a be suspended horizontally by a fine wire attached to the centre, and made to oscillate. Let r be the radius vector of any element of the disk, measured from its centre, θ the angle through which the disk has turned from its mean position. Then in Equation 2.13, we must put $V = r \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$, whence

$$T_3 = \rho \cdot \sqrt{\frac{n\mu'}{2}} \cdot r \cdot \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{1}{n} \cdot \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right) .$$

The area of the annulus of the disk comprised between the radii r and $r + \mathrm{d}r$ is $4\pi r\,\mathrm{d}r$, both faces being taken, and if G be the whole moment of the force of the fluid on the disk,

$$G = -4\pi \cdot \int_0^a r^2 T_3\,\mathrm{d}r\,,$$

whence

$$G = -\pi\rho a^4 \cdot \sqrt{\frac{n\mu'}{2}} \cdot \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{1}{n} \cdot \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right).$$

Let $M\gamma^2$ be the moment of inertia of the disk, and let n_1 be what n would become if the fluid were removed, so that $-n_1^2 M\gamma^2\theta$ is the moment of the force of torsion. Then when the fluid is present the equation of motion of the disk becomes

$$\left\{M\gamma^2 + \pi\rho a^4 \cdot \sqrt{\frac{\mu'}{2n}}\right\} \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \pi\rho a^4 \cdot \sqrt{\frac{n\mu'}{2}} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} + n_1^2 M\gamma^2\theta = 0\,, \tag{2.14}$$

or, putting for shortness

$$\pi\rho a^4 \cdot \sqrt{\frac{\mu'}{2n}} = 2\beta M\gamma^2\,,$$

Equation 2.14 becomes

$$(1 + 2\beta) \cdot \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2n\beta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} + n_1^2\theta = 0\,,$$

which gives, neglecting β^2,

$$\theta = \theta_0 \cdot e^{-n\beta t} \cdot \sin(nt + \alpha)\,, \tag{2.15}$$

where

$$n = n_1 \cdot (1 - \beta)\,.$$

The observation of n and n_1 or else the observation of n and of the decrement of the arc of oscillation, would enable us to determine β, and thence μ'. The values of β determined in these two different ways ought to agree.

There would be no difficulty in obtaining a more exact solution, in which the decrement of the arc of oscillation should be taken into account in calculating the motion of the fluid, but I pass on to the problems, the solution of which forms the main object of this paper.

2.2. Solution of the equations in the case of a sphere oscillating in a mass of fluid either unlimited, or confined by a spherical envelope concentric with the sphere in its position of equilibrium

9. Suppose the sphere suspended by a fine wire, the length of which is much greater than the radius of the sphere. Neglect for the present the action of the wire on the fluid, and consider only that of the sphere. The motion of the sphere and wire being supposed to take place parallel to a fixed vertical plane, there are two different modes of oscillation possible. We have here nothing to do with the rapid oscillations which depend mainly on the rotatory inertia of the sphere, but only with the principal oscillations, which are those which are observed in pendulum experiments. In these principal oscillations the centre of the sphere describes a small arc of a curve which is very nearly a circle, and which would be rigorously such, if the line joining the centre of gravity of the sphere and the point of attachment of the wire were rigorously in the direction of the wire. In calculating the motion of the fluid, we may regard this arc as a right line. In fact, the error thus introduced would only be a small quantity of the second order, and such quantities are supposed to be neglected in the investigation. Besides its motion of translation, the sphere will have a motion of rotation about a horizontal axis, the angular motion of the sphere being very nearly the same as that of the suspending wire. This motion, which would produce absolutely no effect on the fluid according to the common theory of hydrodynamics, will not be without its influence when friction is taken into account; but the effect is so very small in practical cases that it is not worth while to take it into account. For if a be the radius of the sphere, and l the length of the suspending wire, the velocity of a point in the surface of the sphere due to the motion of rotation will be a small quantity of the order $\frac{a}{l}$ compared with the velocity due to the motion of translation. In finding the moment of the pressures of the fluid on the pendulum, forces arising from these velocities, and comparable with them, have to be multiplied by lines which are comparable with a and l respectively. Hence the moment of the pressures due to the motion of rotation of the sphere will be a small quantity of the order $\frac{a^2}{l^2}$, compared with the moment due to the motion of translation. Now in practice l is usually at least 20 or 30 times greater than a, and the whole effect to be investigated is very small, so that it would be quite useless to take account of the motion of rotation of the sphere.

The problem, then, reduces itself to this:

> The centre of a sphere performs small periodic oscillations along a right line, the sphere itself having a motion of translation simply: it is required to determine the motion of the surrounding fluid.

10. Let the mean position of the centre of the sphere be taken for origin, and the direction of its motion for the axis of x, so that the motion of the fluid is symmetrical with respect to this axis. Let ϖ be the perpendicular let fall from any point on the axis of x, q the velocity in the direction of ϖ, ω the angle between the line ϖ and the plane of xy. Then p, u, and q will be functions of x, ϖ, and t, and we shall have

$$\begin{aligned} v &= q \cdot \cos(\omega)\,, \quad & w &= q \cdot \sin(\omega) \\ y &= \varpi \cdot \cos(\omega) & z &= \varpi \cdot \sin(\omega)\,, \end{aligned}$$

whence

$$\begin{aligned} \varpi^2 &= y^2 + z^2 \\ \omega &= \frac{1}{\tan\left(\frac{z}{y}\right)}\,. \end{aligned}$$

We have now to substitute in Equation 2.2 and Equation 2.3, and we are at liberty to put $\omega = 0$ after differentiation. We get

$$\frac{\mathrm{d}}{\mathrm{d}y} = \cos(\omega) \cdot \frac{\mathrm{d}}{\mathrm{d}\varpi} - \frac{\sin(\omega)}{\varpi} \cdot \frac{\mathrm{d}}{\mathrm{d}\omega}\,, \; = \frac{\mathrm{d}}{\mathrm{d}\varpi} \quad \text{when} \quad \omega = 0\,,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} = \frac{\mathrm{d}^2}{\mathrm{d}\varpi^2} \quad \text{when} \quad \omega = 0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}z} = \sin(\omega) \cdot \frac{\mathrm{d}}{\mathrm{d}\varpi} - \frac{\cos(\omega)}{\varpi} \cdot \frac{\mathrm{d}}{\mathrm{d}\omega}\,, \; = \frac{1}{\varpi} \cdot \frac{\mathrm{d}}{\mathrm{d}\omega} \quad \text{when} \quad \omega = 0\,,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} = \frac{1}{\varpi} \cdot \frac{\mathrm{d}}{\mathrm{d}\varpi} + \frac{1}{\varpi^2} \cdot \frac{\mathrm{d}^2}{\mathrm{d}\omega^2} \quad \text{when} \quad \omega = 0\,,$$

whence we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \mu \cdot \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2u}{\mathrm{d}\varpi^2} + \frac{1}{\varpi} \cdot \frac{\mathrm{d}u}{\mathrm{d}\varpi}\right) - \rho \cdot \frac{\mathrm{d}u}{\mathrm{d}t}\,, \tag{2.16}$$

$$\frac{\mathrm{d}p}{\mathrm{d}\varpi} = \mu \cdot \left(\frac{\mathrm{d}^2q}{\mathrm{d}x^2} + \frac{\mathrm{d}^2q}{\mathrm{d}\varpi^2} + \frac{1}{\varpi} \cdot \frac{\mathrm{d}q}{\mathrm{d}\varpi} - \frac{q}{\varpi^2}\right) - \rho \cdot \frac{\mathrm{d}q}{\mathrm{d}t}\,, \tag{2.17}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}q}{\mathrm{d}\varpi} + \frac{q}{\varpi} = 0\,. \tag{2.18}$$

Eliminating p from Equation 2.16 and Equation 2.17, and putting for μ its equivalent $\mu'\rho$, we get

$$\mu' \cdot \frac{\mathrm{d}}{\mathrm{d}\varpi}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}^2}{\mathrm{d}\varpi^2}+\frac{1}{\varpi}\cdot\frac{\mathrm{d}}{\mathrm{d}\varpi}\right)\cdot u-\mu'\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}^2}{\mathrm{d}\varpi^2}+\frac{1}{\varpi}\cdot\frac{\mathrm{d}}{\mathrm{d}\varpi}-\frac{1}{\varpi^2}\right)\cdot q$$
$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u}{\mathrm{d}\varpi}-\frac{\mathrm{d}q}{\mathrm{d}x}\right)=0\,,$$

or

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}^2}{\mathrm{d}\varpi^2}+\frac{1}{\varpi}\cdot\frac{\mathrm{d}}{\mathrm{d}\varpi}-\frac{1}{\varpi^2}-\frac{1}{\mu'}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\right)\cdot\left(\frac{\mathrm{d}u}{\mathrm{d}\varpi}-\frac{\mathrm{d}q}{\mathrm{d}x}\right)=0\,. \tag{2.19}$$

By virtue of Equation 2.18, $\varpi(u\,\mathrm{d}\varpi - q\,\mathrm{d}x)$ is an exact differential. Let then

$$\varpi(u\,\mathrm{d}\varpi - q\,\mathrm{d}x)=\mathrm{d}\psi\,. \tag{2.20}$$

Expressing u and q in terms of ψ, we get

$$\frac{\mathrm{d}u}{\mathrm{d}\varpi}-\frac{\mathrm{d}q}{\mathrm{d}x}=\frac{1}{\varpi}\cdot\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}^2}{\mathrm{d}\varpi^2}-\frac{1}{\varpi}\cdot\frac{\mathrm{d}}{\mathrm{d}\varpi}\right)\psi\,.$$

Substituting in Equation 2.19, and operating separately on the factor $\frac{1}{\varpi}$, we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}^2}{\mathrm{d}\varpi^2}-\frac{1}{\varpi}\cdot\frac{\mathrm{d}}{\mathrm{d}\varpi}-\frac{1}{\mu'}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\right)\cdot\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}^2}{\mathrm{d}\varpi^2}-\frac{1}{\varpi}\cdot\frac{\mathrm{d}}{\mathrm{d}\varpi}\right)\psi=0\,. \tag{11.20'}$$

Since the operations represented by the two expressions within parentheses are

evidently convertible, the integral of this equation is[4]

$$\psi = \psi_1 + \psi_2 \,, \tag{2.21}$$

where ψ_1 and ψ_2 are the integrals of the equations

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}\varpi^2} - \frac{1}{\varpi}\cdot\frac{\mathrm{d}}{\mathrm{d}\varpi}\right)\psi_1 = 0\,, \tag{2.22}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}\varpi^2} - \frac{1}{\varpi}\cdot\frac{\mathrm{d}}{\mathrm{d}\varpi} - \frac{1}{\mu'}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\right)\psi_2 = 0\,. \tag{2.23}$$

11. By means of the last three equations, the expression for $\mathrm{d}p$ obtained from Equation 2.16 and Equation 2.17 is greatly simplified. We get, in the first place,

$$\frac{1}{\rho}\cdot\frac{\mathrm{d}p}{\mathrm{d}x} = \left\{\mu'\cdot\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}\varpi^2} + \frac{1}{\varpi}\cdot\frac{\mathrm{d}}{\mathrm{d}\varpi}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\right\}\cdot\frac{1}{\varpi}\cdot\frac{\mathrm{d}\psi}{\mathrm{d}\varpi}\ : \tag{2.24}$$

[4]If we denote for shortness the operation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}\varpi^2} - \frac{1}{\varpi}\cdot\frac{\mathrm{d}}{\mathrm{d}\varpi}$$

by D, our equation becomes

$$\mathrm{D}\left(\mathrm{D} - \frac{1}{\mu'}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\right)\psi = 0\,,$$

which gives by the separation of symbols

$$\begin{aligned}\psi &= \frac{1}{\mathrm{D}\left(\mathrm{D} - \frac{1}{\mu'}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\right)}\,0 \\ &= \frac{1}{\frac{1}{\mu'}\cdot\frac{\mathrm{d}}{\mathrm{d}t}}\cdot\left\{\frac{1}{\mathrm{D} - \frac{1}{\mu'}\cdot\frac{\mathrm{d}}{\mathrm{d}t}} - \frac{1}{\mathrm{D}}\right\}0\,,\end{aligned} \tag{β}$$

so that $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ is composed of two parts, which are separately the integrals of Equation 2.22 and Equation 2.23. Hence we have for the integral of Equation 11.20' $\psi = \psi_1 + \psi_2 + \Psi$, Ψ being a function of x and ϖ without t, which satisfies the equation $D^2\Psi = 0$. For the Equation 2.22 and Equation 2.23 will not be altered if we put $\int \psi_1\,\mathrm{d}t$, $\int \psi_2\,\mathrm{d}t$ for ψ_1 and ψ_2, the arbitrary functions which would arise from the integration with respect to t being supposed to be included in Ψ. The function Ψ, which taken by itself can only correspond to steady motion, is excluded from the problem under consideration by the condition of periodicity. But we may even, independently of this condition, regard Equation 2.21 as the complete integral of Equation 11.20', provided we suppose included in Equation 2.21 terms which would be obtained by supposing ψ at first to vary slowly with the time, employing the integrals of Equation 2.22 and Equation 2.23, and then making the rate of variation diminish indefinitely. By treating the symbolical expression in the right-hand member of Equation β as a vanishing fraction, $\frac{\mathrm{d}}{\mathrm{d}t}$ being supposed to vanish, we obtain in fact $\mathrm{D}^{-2}0$; so that under the convention just mentioned the function Ψ may be supposed to be included in $\psi_1 + \psi_2$. The same remarks will apply to the equation in section 2.3 which answers to Equation 11.20'.

but by adding together Equation 2.22 and Equation 2.23, and taking account of Equation 2.21, we get

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -\frac{\mathrm{d}^2\psi}{\mathrm{d}\varpi^2} + \frac{1}{\varpi}\cdot\frac{\mathrm{d}\psi}{\mathrm{d}\varpi} + \frac{1}{\mu'}\cdot\frac{\mathrm{d}\psi_2}{\mathrm{d}t}\,.$$

On substituting in Equation 2.24, it will be found that all the terms in the right-hand member of the equation destroy one another, except those which contain $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ and $\frac{\mathrm{d}\psi_2}{\mathrm{d}t}$, and the equation is reduced to

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{\rho}{\varpi}\cdot\frac{\mathrm{d}^2\psi_1}{\mathrm{d}t\,\mathrm{d}\varpi}\,.$$

The Equation 2.17 may be reduced in a similar manner, and we finally get

$$\mathrm{d}p = \frac{\rho}{\varpi}\cdot\left\{\frac{\mathrm{d}^2\psi_1}{\mathrm{d}t\,\mathrm{d}x}\,\mathrm{d}\varpi - \frac{\mathrm{d}^2\psi_1}{\mathrm{d}t\,\mathrm{d}\varpi}\,\mathrm{d}x\right\}, \tag{2.25}$$

which is an exact differential by virtue of Equation 2.22.

12. Passing to polar co-ordinates, let r be the radius vector drawn from the origin, θ the angle which r makes with the axis of x, and let R be the velocity along the radius vector, Θ the velocity perpendicular to the radius vector: then

$$\begin{aligned} x &= r\cdot\cos(\theta)\,,\\ \varpi &= r\cdot\sin(\theta)\,,\\ u &= R\cdot\cos(\theta) - \Theta\cdot\sin(\theta)\,,\\ q &= R\cdot\sin(\theta) + \Theta\cdot\cos(\theta)\,. \end{aligned}$$

Making these substitutions in Equation 2.20, Equation 2.22, Equation 2.23, and Equation 2.25, we obtain

$$r\cdot\sin(\theta)\cdot(Rr\,\mathrm{d}\theta - \Theta\,\mathrm{d}r) = \mathrm{d}\psi\,, \tag{2.26}$$

$$\frac{\mathrm{d}^2\psi_1}{\mathrm{d}r^2} + \frac{\sin(\theta)}{r^2}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\sin(\theta)}\cdot\frac{\mathrm{d}\psi_1}{\mathrm{d}\theta}\right) = 0\,, \tag{2.27}$$

$$\frac{\mathrm{d}^2\psi_2}{\mathrm{d}r^2} + \frac{\sin(\theta)}{r^2}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\sin(\theta)}\cdot\frac{\mathrm{d}\psi_2}{\mathrm{d}\theta}\right) - \frac{1}{\mu'}\cdot\frac{\mathrm{d}\psi_2}{\mathrm{d}t} = 0\,, \tag{2.28}$$

$$\mathrm{d}p = \frac{\rho}{r\cdot\sin(\theta)}\cdot\left(\frac{\mathrm{d}^2\psi_1}{\mathrm{d}t\,\mathrm{d}r}\cdot r\,\mathrm{d}\theta - \frac{1}{r}\cdot\frac{\mathrm{d}^2\psi_1}{\mathrm{d}t\,\mathrm{d}\theta}\,\mathrm{d}r\right). \tag{2.29}$$

We must now determine ψ_1 and ψ_2 by means of Equation 2.27 and Equation 2.28, combined with the equations of condition. When these functions are known, p will be obtained by integrating the exact differential which forms the right-hand

member of Equation 2.29, and the velocities R, Θ, if required, will be got by differentiation, as indicated by Equation 2.26. Formulæ deduced from Equation 2.4 and Equation 2.5 will then make known the pressure of the fluid on the sphere.

13. Let ξ be the abscissa of the centre of the sphere at any instant. The conditions to be satisfied at the surface of the sphere are that when $r = r_1$, the radius vector of the surface, we have

$$R = \cos(\theta) \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t}\,, \quad \Theta = -\sin(\theta) \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t}\,.$$

Now r_1 differs from a by a small quantity of the first order, and since this value of r has to be substituted in functions which are already small quantities of that order, it will be sufficient to put $r = a$. Hence, expressing R and Θ in terms of ψ, we get

$$\left.\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}r} &= a \cdot \sin^2(\theta) \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t}\,, \\ \frac{\mathrm{d}\psi}{\mathrm{d}\theta} &= a^2 \cdot \sin(\theta)\cos(\theta) \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t}\,, \end{aligned}\right\} \tag{2.30}$$

when $r = a$. When the fluid is unlimited, it will be found that certain arbitrary constants will vanish by the condition that the motion shall not become infinite at an infinite distance in the fluid. When the fluid is confined by an envelope having a radius b, we have the equations of condition

$$\frac{\mathrm{d}\psi}{\mathrm{d}r} = 0\,, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\theta} = 0\,, \qquad \text{when} \quad r = b\,. \tag{2.31}$$

14. We must now, in accordance with the plan proposed in section 2.1, introduce the condition that the function ψ shall be composed, so far as the time is concerned, of the circular functions $\sin(nt)$ and $\cos(nt)$, that is, that it shall be of the form

$$P \cdot \sin(nt) + Q \cdot \cos(nt)$$

where P and Q are functions of r and θ only. An artifice, however, which has been extensively employed by M. Cauchy will here be found of great use. Instead of introducing the circular functions $\sin(nt)$ and $\cos(nt)$, we may employ the exponentials $e^{\mathbf{i}\,nt}$, and $e^{-\mathbf{i}\,nt}$, where $\mathbf{i} = \sqrt{-1}$. Since our equations are linear, and since each of these exponential functions reproduces itself at each differentiation, it follows that if all the terms in any one of our equations be arranged in two groups, containing as a factor $e^{\mathbf{i}\,nt}$ in one case, and $e^{-\mathbf{i}\,nt}$ in the other, the two groups will be quite independent, and the equations will be satisfied by either group separately. Hence it will be sufficient to introduce one of the exponential functions. We shall thus have only half the number of terms to write down, and half the number of arbitrary constants to determine that would have been necessary had we employed

circular functions. When we have arrived at our result, it will be sufficient to put each equation under the form $U + \mathbf{i}\,V = 0$, and throw away the imaginary part, or else throw away the real part and omit $\mathbf{i}$ since the system of quantities U, and the system of quantities V must separately satisfy the equations of the problem. Assuming then

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = c \cdot e^{\mathbf{i}\,nt}\,,$$
$$\psi = e^{\mathbf{i}\,nt} \cdot P\,,$$

we have to determine P as a function of r and θ.

15. The form of the equations of condition according to Equation 2.30 points out $\sin^2(\theta)$ as a factor of P, and since the operation $\sin(\theta) \cdot \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\sin(\theta)}\frac{\mathrm{d}}{\mathrm{d}\theta}$ performed on the function $\sin^2(\theta)$ reproduces the same function with a coefficient -2, it will be possible to satisfy Equation 2.27 and Equation 2.28 on the supposition that $\sin^2(\theta)$ is a factor of ψ_1 and ψ_2[5]. Assume then

$$\psi_1 = e^{\mathbf{i}\,nt} \cdot \sin^2(\theta) \cdot f_1(r)\,,$$
$$\psi_2 = e^{\mathbf{i}\,nt} \cdot \sin^2(\theta) \cdot f_2(r)\,.$$

Putting for convenience

$$n\,\mathbf{i} = \mu' \cdot m^2\,, \tag{2.32}$$

and substituting in Equation 2.27 and Equation 2.28, we get

$$f_1''(r) - \frac{2}{r^2} \cdot f_1(r) = 0\,, \tag{2.33}$$

[5]When this operation is performed on the function $\sin(\theta)\frac{\mathrm{d}Y_i}{\mathrm{d}\theta}$, the function is reproduced with a coefficient $-i \cdot (i+1)$. Y_i here denotes a Laplace's coefficient of the i^{th} order, which contains only one variable angle, namely θ. Now ψ may be expanded in a series of quantities of the general form $\sin(\theta)\frac{\mathrm{d}Y_i}{\mathrm{d}\theta}$. For, since we are only concerned with the differential coefficients of ψ with respect to r and θ, we have a right to suppose ψ to vanish at whatever point of space we please. Let then $\psi = 0$ when $r = a$ and $\theta = 0$. To find the value of ψ at a distance r from the origin, along the axis of x positive, it will be sufficient to put $\theta = 0$ and $\mathrm{d}\theta = 0$ in Equation 2.26, and integrate from $r = a$ to r, whence $\psi = 0$. To find the value of ψ at the same distance r along the axis of x negative, it will be sufficient to leave r constant, and integrate $\mathrm{d}\psi$ from $\theta = 0$ to $\theta = \pi$. Referring to Equation 2.26, we see that the integral vanishes, since the total flux across the surface of the sphere whose radius is r must be equal to zero. Hence ψ vanishes when $\theta = 0$ or $\theta = \pi$, and it appears from Equation 2.26 that when θ is very small or very nearly equal to π, ψ varies ultimately as $\sin^2(\theta)$ for given values of r and t. Hence $\psi \cdot \operatorname{cosec}\theta$, and therefore $\int \psi \cdot \operatorname{cosec}(\theta)\,\mathrm{d}\theta$, is finite even when $\sin(\theta)$ vanishes, and therefore $\int \psi \cdot \operatorname{cosec}(\theta)\,\mathrm{d}\theta$ may be expanded in a series of Laplace's coefficients, and therefore ψ itself in a series of quantities of the form $\sin(\theta)\frac{\mathrm{d}Y_i}{\mathrm{d}\theta}$. It was somewhat in this way that I first obtained the form of the function ψ.

$$f_2''(r) - \frac{2}{r^2} \cdot f_2(r) - m^2 \cdot f_2(r) = 0\,. \tag{2.34}$$

The equations of condition, Equation 2.30 and Equation 2.31 become, on putting $f(r)$ for $f_1(r) + f_2(r)$,

$$f'(a) = a \cdot c\,, \quad f(a) = \frac{1}{2} \cdot a^2 \cdot c\,, \tag{2.35}$$

$$f'(b) = 0\,, \qquad f(b) = 0\,. \tag{2.36}$$

We may obtain p from Equation 2.29 by putting for ψ_1 its value $e^{\mu' m^2 t} \cdot \sin^2(\theta) \cdot f_1(r)$, replacing after differentiation $2 \cdot f_1(r)$ by its equivalent $r^2 \cdot f_1''(r)$ and then integrating. It is unnecessary to add an arbitrary function of the time, since any such function may be supposed to be included in Π. We get

$$p = -\rho\mu' m^2 \cdot e^{\mu' m^2 t} \cdot \cos(\theta) \cdot f_1'(r)\,. \tag{2.37}$$

16. The integration of the differential Equation 2.33 does not present the least difficulty, and Equation 2.34 comes under a well-known integrable form. The integrals of these equations are

$$\left.\begin{aligned} f_1(r) &= \frac{A}{r} + B \cdot r^2\,, \\ f_2(r) &= C \cdot e^{-mr} \cdot \left(1 + \frac{1}{mr}\right) + D \cdot e^{mr} \cdot \left(1 - \frac{1}{mr}\right)\,, \end{aligned}\right\} \tag{2.38}$$

and we have to determine A, B, C, D by the equations of condition.
The solution of the problem, in the case in which the fluid is confined by a spherical envelope, will of course contain as a particular case that in which the fluid is unlimited, to obtain the results belonging to which it will be sufficient to put $b = \infty$. As, however, the case of an unlimited fluid is at the same time simpler and more interesting than the general case, it will be proper to consider it separately. Let $+m$ denote that square root of $\frac{n}{\mu'}\,\mathbf{i}$ which has its real part positive; then in Equation 2.38 we must have $D = 0$, since otherwise the velocity would be infinite at an infinite distance. We must also have $B = 0$, since otherwise the velocity would be finite when $r = \infty$, as appears from Equation 2.26. We get then from the equations of condition, Equation 2.35

$$\begin{aligned} A &= \frac{1}{2}a^3 c + \frac{3a^2 c}{2m} \cdot \left(1 + \frac{1}{ma}\right)\,, \\ C &= -\frac{3ac}{2m} \cdot e^{ma}\,, \end{aligned}$$

whence

$$\xi = \frac{c}{\mu' m^2} \cdot e^{\mu' m^2 t}\,, \tag{2.39}$$

$$\boxed{\psi = \frac{1}{2} \cdot a^2 c \cdot e^{\mu' m^2 t} \cdot \sin^2(\theta) \cdot \left\{ \left(1 + \frac{3}{ma} + \frac{3}{m^2 a^2} \right) \cdot \frac{a}{r} - \frac{3}{ma} \cdot \left(1 + \frac{1}{mr} \right) \cdot e^{-m \cdot (r-a)} \right\}} \tag{2.40}$$

$$\boxed{p = \frac{1}{2} \cdot \rho a c \cdot \mu' m^2 \cdot \left(1 + \frac{3}{ma} + \frac{3}{m^2 a^2} \right) \cdot e^{\mu' m^2 t} \cdot \cos(\theta) \cdot \frac{a^2}{r^2}} \tag{2.41}$$

17. The symbolical Equation 2.40 and Equation 2.41 contain the solution of the problem, the motion of the sphere being defined by the symbolical Equation 2.39. If we wish to exhibit the actual results by means of real quantities alone, we have only to put the right-hand members of Equation 2.39, Equation 2.40, Equation 2.41 under the form $U + \mathbf{i}\, V$, and reject the imaginary part. Putting for shortness

$$\sqrt{\frac{n}{2 \cdot \mu'}} = \nu \,, \tag{2.42}$$

we have $m = \nu \cdot (1 + \mathbf{i})$, and we obtain

$$\xi = \frac{c}{n} \cdot \sin(nt) \,, \tag{2.43}$$

$$\begin{aligned} \psi = \frac{1}{2} \cdot a^2 c \cdot \sin(\theta) \cdot \Bigg\{ \Bigg[\left(1 + \frac{3}{2\nu a} \right) \cdot \cos(nt) \\ + \frac{3}{2\nu a} \cdot \left(1 + \frac{1}{\nu a} \right) \cdot \sin(nt) \Bigg] \cdot \frac{a}{r} \\ - \frac{3}{2\nu a} \cdot e^{-\nu \cdot (r-a)} \cdot \Big[\cos(nt - \nu r + \nu a) \\ + \left(1 + \frac{1}{\nu r} \right) \cdot \sin(nt - \nu r + \nu a) \Bigg] \Bigg\} \,, \end{aligned} \tag{2.44}$$

$$\begin{aligned} p = -\frac{1}{2} \cdot \rho a c n \cdot \Bigg\{ \left(1 + \frac{3}{2\nu a} \right) \cdot \sin(nt) \\ - \frac{3}{2\nu a} \cdot \left(1 + \frac{1}{\nu a} \right) \cdot \cos(nt) \Bigg\} \cdot \cos(\theta) \cdot \frac{a^2}{r^2} \,. \end{aligned} \tag{2.45}$$

The reader will remark that the ξ, ψ, p of the present article are not the same as the ξ, ψ, p of the preceding. The latter are the imaginary expressions, of which the real parts constitute the former. It did not appear necessary to change the notation.

When $\mu' = 0$, then $\nu = \infty$ and ψ reduces itself to

$$\frac{a^3 c}{2r} \cdot \sin^2(\theta) \cos(nt)\,,$$

or

$$\frac{a^3}{2r} \cdot \sin^2(\theta) \frac{\mathrm{d}\xi}{\mathrm{d}t}\,.$$

In this case we get from Equation 2.26

$$R = a^3 \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t} \cdot \frac{\cos(\theta)}{r^3}\,,$$
$$\Theta = \frac{1}{2} a^3 \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t} \cdot \frac{\sin(\theta)}{r^3}\,,$$

and $R\,\mathrm{d}r + \Theta r\,\mathrm{d}\theta$ is an exact differential $\mathrm{d}\phi$, where

$$\phi = -\frac{1}{2} a^3 \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t} \cdot \frac{\cos(\theta)}{r^2}\,,$$

which agrees with the result deduced directly from the ordinary equations of hydrodynamics[6].

18. Let us now form the expression for the resultant of the pressures of the fluid on the several elements of the surface of the sphere. Let P_r be the normal, and T_θ the tangential component of the pressure at any point in the direction of a plane drawn perpendicular to the radius vector. Equation 2.4 and Equation 2.5 are general, and therefore we may replace x, y in these formulæ by x', y', where x', y' are measured in any two rectangular directions we please. Let the plane of x', y' pass through the axis of x and the radius vector, and let the axis of x' be inclined to that of x at an angle ϑ, which after differentiation is made equal to θ. Then P_1 and T_3 will become P_r and T_θ, respectively. We have

$$u' = R \cdot \cos(\theta - \vartheta) - \Theta \cdot \sin(\theta - \vartheta)\,,$$
$$v' = R \cdot \sin(\theta - \vartheta) + \Theta \cdot \cos(\theta - \vartheta)\,,$$

[6]See *Camb. Phil. Trans.* Vol. VIII., page 39.

and when $\theta = \vartheta$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x'} &= \frac{\mathrm{d}}{\mathrm{d}r}\,, \\
\frac{\mathrm{d}}{\mathrm{d}y'} &= \frac{1}{r}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\,, \\
\frac{\mathrm{d}u'}{\mathrm{d}x'} &= \frac{\mathrm{d}R}{\mathrm{d}r}\,, \\
\frac{\mathrm{d}u'}{\mathrm{d}y'} &= \frac{1}{r}\cdot\frac{\mathrm{d}R}{\mathrm{d}\theta} - \frac{\Theta}{r}\,, \\
\frac{\mathrm{d}v'}{\mathrm{d}x'} &= \frac{\mathrm{d}\Theta}{\mathrm{d}r}\,;
\end{aligned}$$

whence

$$\left.\begin{aligned}
P_r &= p - 2\mu\cdot\frac{\mathrm{d}R}{\mathrm{d}r}\,, \\
T_\theta &= -\mu\cdot\left(\frac{1}{r}\cdot\frac{\mathrm{d}R}{\mathrm{d}\theta} + \frac{\mathrm{d}\Theta}{\mathrm{d}r} - \frac{\Theta}{r}\right)\,.
\end{aligned}\right\} \tag{2.46}$$

In these formulæ, suppose r put equal to a after differentiation. Then P_r and T_θ will be the components in the direction of r and θ of the pressure of the sphere on the fluid. The resolved part of these in the direction of x is

$$P_r\cdot\cos(\theta) - T_\theta\cdot\sin(\theta)\,,$$

which is equal and opposite to the component, in the direction of x, of the pressure of the fluid on the sphere. Let $\mathscr{F}$ be the whole force of the fluid on the sphere, which will evidently act along the axis of x. Then, observing that $2\pi a^2\sin(\theta)\,\mathrm{d}\theta$ is the area of an elementary annulus of the surface of the sphere, we get

$$\mathscr{F} = 2\pi a^2\cdot\int_0^\pi \left[-P_r\cdot\cos(\theta) + T_\theta\cdot\sin(\theta)\right]_a \sin(\theta)\,\mathrm{d}\theta\,, \tag{2.47}$$

the suffix a denoting that r is supposed to have the value a in the general expressions for P_r and T_θ.
The expression for $\mathscr{F}$ may be greatly simplified, without employing the solution of Equation 2.27 and Equation 2.28, by combining these equations in their original state with the equations of condition according to Equation 2.30. We have, in the first place, from Equation 2.26

$$\left.\begin{aligned}
R &= \frac{1}{r^2\cdot\sin(\theta)}\cdot\frac{\mathrm{d}\psi}{\mathrm{d}\theta}\,, \\
\Theta &= -\frac{1}{r\cdot\sin(\theta)}\cdot\frac{\mathrm{d}\psi}{\mathrm{d}r}\,.
\end{aligned}\right\} \tag{2.48}$$

Now the Equation 2.30 make known the values of ψ and $\frac{\mathrm{d}\psi}{\mathrm{d}r}$, and of their differential coefficients of all orders with respect to θ, when $r = a$. When the expressions for R and Θ are substituted in Equation 2.46, the result will contain only one term in which the differentiation with respect to r rises to the second order. But we get from Equation 2.21, Equation 2.27, Equation 2.28

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} = -\frac{\sin(\theta)}{r^2} \cdot \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\sin(\theta)} \cdot \frac{\mathrm{d}\psi}{\mathrm{d}\theta}\right) + \frac{1}{\mu'} \cdot \frac{\mathrm{d}\psi_2}{\mathrm{d}t}\,,$$

and the second of Equation 2.30 gives the value for $r = a$ of the first term in the right-hand member of the equation just written. We obtain from Equation 2.48 and Equation 2.30

$$\begin{aligned}\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)_a &= 0\,,\\ \left(\frac{1}{r} \cdot \frac{\mathrm{d}R}{\mathrm{d}\theta}\right)_a &= -\frac{\sin(\theta)}{a} \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t} = \left(\frac{\Theta}{r}\right)_a\,,\\ \left(\frac{\mathrm{d}\Theta}{\mathrm{d}r}\right)_a &= -\frac{1}{\mu' a \cdot \sin(\theta)} \cdot \left(\frac{\mathrm{d}\psi_2}{\mathrm{d}t}\right)_a\,.\end{aligned}$$

Substituting in Equation 2.47, and writing $\mu'\rho$ for μ, we get

$$\mathscr{F} = 2\pi a \cdot \int_0^\pi \left\{-a p_a \cdot \cos(\theta) + \rho \cdot \left(\frac{\mathrm{d}\psi_2}{\mathrm{d}t}\right)_a\right\} \cdot \sin(\theta)\,\mathrm{d}\theta\,.$$

With respect to the first term in this expression, we get by integration by parts

$$\int p \cos(\theta)\sin(\theta)\,\mathrm{d}\theta = \frac{1}{2} \cdot \sin^2(\theta) \cdot p - \frac{1}{2} \cdot \int \sin^2(\theta)\frac{\mathrm{d}p}{\mathrm{d}\theta}\,\mathrm{d}\theta\,.$$

The first term vanishes at the limits. Substituting in the second term for $\frac{\mathrm{d}p}{\mathrm{d}\theta}$ the expression got from Equation 2.29, and putting $r = a$, we get

$$\int_0^\pi p_a \cdot \cos(\theta)\sin(\theta)\,\mathrm{d}\theta = -\frac{1}{2} \cdot \rho \cdot \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\pi \left(\frac{\mathrm{d}\psi_1}{\mathrm{d}r}\right)_a \cdot \sin(\theta)\,\mathrm{d}\theta\,.$$

Substituting in the expression for $\mathscr{F}$, we get

$$\mathscr{F} = \pi\rho a \cdot \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\pi \left\{a \cdot \left(\frac{\mathrm{d}\psi_1}{\mathrm{d}r}\right)_a + 2 \cdot (\psi_2)_a\right\} \cdot \sin(\theta)\,\mathrm{d}\theta\,. \tag{2.49}$$

19. The above expression for $\mathscr{F}$, being derived from the general Equation 2.27 and Equation 2.28, combined with the equations of condition according to Equation 2.30, holds good, not merely when the fluid is confined by a spherical envelope, but whenever the motion is symmetrical about an axis, and that, whether the motion of the sphere be or be not expressed by a single circular function of the time. It might be employed, for instance, in the case of a sphere oscillating in a direction perpendicular to a fixed rigid plane.
When the fluid is either unconfined, or confined by a spherical envelope concentric with the sphere in its position of equilibrium, the functions ψ_1 and ψ_2 consist, as we have seen, of $\sin^2(\theta)$ multiplied by two factors independent of θ. If we continue to employ the symbolical expressions, which will be more convenient to work with than the real expressions which might be derived from them, we shall have

$$e^{\mathbf{i}nt} \cdot f_1(r)\,, \quad e^{\mathbf{i}nt} \cdot f_2(r)\,,$$

for these factors respectively. Substituting in Equation 2.49, and performing the integration with respect to θ, we get

$$\mathscr{F} = \frac{4}{3} \cdot \pi\rho a n\,\mathbf{i} \cdot \{a \cdot f_1'(a) + 2 \cdot f_2(a)\} \cdot e^{\mathbf{i}nt}\,. \tag{2.50}$$

20. Consider for the present only the case in which the fluid is unlimited. The arbitrary constants which appear in Equation 2.38 were determined for this case in Art. 16, page 152. Substituting in Equation 2.50 we get

$$\mathscr{F} = -\frac{2}{3} \cdot \pi\rho a^3 c n\,\mathbf{i} \cdot \left(1 + \frac{9}{ma} + \frac{9}{m^2a^2}\right) \cdot e^{\mathbf{i}nt}\,.$$

Putting for m its value $\nu \cdot (1+\mathbf{i})$, and denoting by M' the mass of the fluid displaced by the sphere, which is equal to $\frac{4}{3}\pi\rho a^3$, we get

$$\mathscr{F} = -M' c n \cdot \left\{\left(\frac{1}{2} + \frac{9}{4\nu a}\right)\mathbf{i} + \frac{9}{4\nu a} \cdot \left(1 + \frac{1}{\nu a}\right)\right\} \cdot e^{\mathbf{i}nt}\,;$$

whence

$$\mathscr{F} = -\left(\frac{1}{2} + \frac{9}{4\nu a}\right) M' \cdot \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} - \frac{9}{4\nu a} \cdot \left(1 + \frac{1}{\nu a}\right) M' n \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t}\,. \tag{2.51}$$

Since $\mathbf{i}$ has been eliminated, this equation will remain unchanged when we pass from the symbolical to the real values of $\mathscr{F}$ and ξ.
Let τ be the time of oscillation from rest to rest, so that $n \cdot \tau = \pi$, and put for

shortness k and k' for the coefficients of M' in Equation 2.51; then

$$\left.\begin{aligned} \nu &= \sqrt{\frac{\pi}{2 \cdot \mu' \tau}}\,, \\ k &= \frac{1}{2} + \frac{9}{4\nu a}\,, \\ k' &= \frac{9}{4\nu a} \cdot \left(1 + \frac{1}{\nu a}\right)\,. \end{aligned}\right\} \tag{2.52}$$

The first term in the expression for the force $\mathscr{F}$ has the same effect as increasing the inertia of the sphere. To take account of this term, it will be sufficient to conceive a mass $k \cdot M'$ collected at the centre of the sphere, adding to its inertia without adding to its weight. The main effect of the second term is to produce a diminution in the arc of oscillation: its effect on the time of oscillation would usually be quite insensible, and must in fact be neglected for consistency's sake, because the motion of the fluid was determined by supposing the motion of the sphere permanent, which is only allowable when we neglect the square of the rate of decrease of the arc of oscillation. If we form the equation of motion of the sphere, introducing the force $\mathscr{F}$, and then proceed to integrate the equation, we shall obtain in the integral an exponential $e^{-\delta t}$ multiplying the circular function, δ being half the coefficient of $\frac{\mathrm{d}\xi}{\mathrm{d}t}$ divided by that of $\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}$. Let M be the mass of the sphere, $M\gamma^2$ its moment of inertia about the axis of suspension, then

$$nk'M' \cdot (l+a)^2 = 2\delta \cdot \left\{M\gamma^2 + kM' \cdot (l+a)^2\right\}\,.$$

In considering the diminution of the arc of oscillation, we may put $(l+a)$ for γ. During i oscillations, let the arc of oscillation be diminished in the ratio of A_0 to A_i, then

$$\ln\left(\frac{A_0}{A_i}\right) = i\tau\delta = \frac{\pi \cdot i}{2} \cdot \frac{k'M'}{(M + kM')}\,. \tag{2.53}$$

For a given fluid and a given time of oscillation, both k and k' increase as a decreases. Hence it follows from theory, that the smaller be the sphere, its density being supposed given, the more the time of oscillation is affected, and the more rapidly the arc of oscillation diminishes, the alteration in the rate of diminution of the arc due to an alteration in the radius of the sphere being more conspicuous than the alteration in the time of oscillation.

21. Let us now suppose the fluid confined in a spherical envelope. In this case, we have to determine the four arbitrary constants which appear in Equation 2.38 by the four equations according to Equation 2.35 and Equation 2.36. We get, in the first place,

$$\frac{A}{a} + Ba^2 + Ce^{-ma} \cdot \left(1 + \frac{1}{ma}\right) + De^{ma} \cdot \left(1 - \frac{1}{ma}\right) = \frac{1}{2}a^2 c\,, \tag{2.54}$$

$$-\frac{A}{a}+2Ba^2-Ce^{-ma}\cdot\left(ma+1+\frac{1}{ma}\right)+De^{ma}\cdot\left(ma-1+\frac{1}{ma}\right)=a^2c\,, \quad (2.55)$$

$$\frac{A}{b}+Bb^2+Ce^{-mb}\cdot\left(1+\frac{1}{mb}\right)+De^{mb}\cdot\left(1-\frac{1}{mb}\right)=0\,, \quad (2.56)$$

$$-\frac{A}{b}+2Bb^2-Ce^{-mb}\cdot\left(mb+1+\frac{1}{mb}\right)+De^{mb}\cdot\left(mb-1+\frac{1}{mb}\right)=0\,. \quad (2.57)$$

Putting a^2cK for $a\cdot f_1'(a)+2\cdot f_2(a)$, which is the quantity that we want to find, we get from Equation 2.38 and Equation 2.54

$$K=1-\frac{3\cdot A}{a^3\cdot c}\,. \quad (2.58)$$

Eliminating in succession B from Equation 2.54 and Equation 2.55, from Equation 2.56 and Equation 2.57, and from Equation 2.54 and Equation 2.56, we shall obtain for the determination of A, C, D three equations which remain unchanged when a and b are interchanged, and the signs of A, C, and D changed. Hence $-A$, $-C$, $-D$ are the same functions of b and a that A, C, D are of a and b. It will also assist in the further elimination to observe that C and D are interchanged when the sign of m is changed. The result of the elimination is

$$K=1-\left\{\frac{3b}{2m^2a^2}\cdot\frac{\eta(a,b)-\eta(b,a)}{12mab+\zeta(a,b)+\zeta(b,a)}\right\}\,, \quad (2.59)$$

the functions ζ and η being defined by the equations

$$\left.\begin{aligned}\eta(a,b)&=\left(m^2a^2+3ma+3\right)\cdot\left(m^2b^2-3mb+3\right)\cdot e^{m\cdot(b-a)}\,,\\ \zeta(a,b)&=\left\{b\cdot\left(m^2b^2-3mb+3\right)-a\cdot\left(m^2a^2+3ma+3\right)\right\}\cdot e^{m\cdot(b-a)}\,.\end{aligned}\right\} \quad (2.60)$$

It turns out that K is a complicated function of m and $\frac{a}{b}$, and the algebraical expressions for the quantities which answer to k and k' in Art. 20, page 157, would be more complicated still, because $\nu\cdot(1+\mathbf{i})$ would have to be substituted for m in Equation 2.60 and Equation 2.59, and then K reduces to the form $-k+\mathbf{i}\,k'$. To obtain numerical results from these formulæ, it would be best to substitute the numerical values of a, b, and ν in Equation 2.60 and Equation 2.59, and perform the reduction of K in figures.

22. If the distance of the envelope from the surface of the sphere be at all considerable, the exponential $e^{\nu\cdot(b-a)}$, which arises from $e^{m\cdot(b-a)}$, will have so large a numerical value that we may neglect the terms in the numerator and denominator of the fraction in the expression for K which contain $e^{-\nu\cdot(b-a)}$, as well as the term in the denominator which is free from exponentials, in comparison with the terms which contain $e^{\nu\cdot(b-a)}$. Thus, if $(b-a)$ be two inches, τ one second, and $\sqrt{\mu'}=0.116$, we have $e^{\nu\cdot(b-a)}=2424000000$, nearly; and if $(b-a)$ be only an inch or half an

inch, we have still the square or fourth root of the above quantity, that is, about 49234 or 222, for the value of that exponential. Hence, in practical cases, the above simplification may be made, which will cause the exponentials to disappear from the expression for K. We thus get

$$K = 1 - \left\{ \frac{3b}{2m^2a^2} \cdot \frac{(m^2a^2 + 3ma + 3) \cdot (m^2b^2 - 3mb + 3)}{b \cdot (m^2b^2 - 3mb + 3) - a \cdot (m^2a^2 + 3ma + 3)} \right\} . \qquad (2.61)$$

If we assume

$$\begin{aligned} 3\nu a + 3 + \left(2\nu^2a^2 + 3\nu a\right)\mathbf{i} &= A' \cdot [\cos(\alpha) + \mathbf{i}\sin(\alpha)] \,, \\ -3\nu b + 3 + \left(2\nu^2b^2 - 3\nu b\right)\mathbf{i} &= B' \cdot [\cos(\beta) + \mathbf{i}\sin(\beta)] \,, \\ bB' \cdot \cos(\beta) - aA' \cdot \cos(\alpha) &= C' \cdot \cos(\gamma) \,, \\ bB' \cdot \sin(\beta) - aA' \cdot \sin(\alpha) &= C' \cdot \sin(\gamma) \,, \end{aligned}$$

we get from Equation 2.61

$$K = 1 + \left\{ \frac{3b\,\mathbf{i}}{4\nu^2a^2} \cdot \frac{A'B'}{C'} \cdot [\cos(\alpha + \beta - \gamma) + \mathbf{i}\sin(\alpha + \beta - \gamma)] \right\} ,$$

whence

$$\left.\begin{aligned} k &= \frac{3bA'B'}{4\nu^2a^2C'} \cdot \sin(\alpha + \beta - \gamma) - 1 \,, \\ k' &= \frac{3bA'B'}{4\nu^2a^2C'} \cdot \cos(\alpha + \beta - \gamma) \,; \end{aligned}\right\} \qquad (2.62)$$

and, as before, $k \cdot M'$ is the imaginary mass which we must conceive to be collected at the centre of the sphere, in order to allow for the inertia of the fluid, and $-k'M'n \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t}$ the term in $\mathscr{F}$ on which depends the diminution in the arc of oscillation.

23. If we suppose $\mu' = 0$, and therefore $m = \infty$, we get from Equation 2.61

$$K = -\frac{b^3 + 2a^3}{2 \cdot (b^3 - a^3)} \,, \qquad (2.63)$$

and, in this case, k is the same as K with sign changed, and $k' = 0$, which agrees with the result obtained directly from the ordinary equations of hydrodynamics[7]. If, on the other hand, we make $b = \infty$, we arrive at the results already obtained in Art. 20, page 157. In both these cases it becomes rigorously exact to neglect in the expression for $(K - 1)$ given by Equation 2.59 all the terms which are not multiplied by $e^{\nu \cdot (b-a)}$.
If the effect of the envelope be but small, which will generally be the case, it will

[7]See *Camb. Phil. Trans.* Vol. VIII. p. 39.

be convenient to calculate k and k' from Equation 2.52, which apply to the case in which $b = \infty$, and then add corrections Δk, $\Delta k'$ due to the envelope. We get from Equation 2.61

$$\Delta k - \mathbf{i}\,\Delta k' = \frac{3}{2m^2a} \cdot \frac{(m^2a^2 + 3ma + 3)^2}{b \cdot (m^2b^2 - 3mb + 3) - a \cdot (m^2a^2 + 3ma + 3)}\,, \qquad (2.64)$$

which may be treated, if required, as the Equation 2.61 was treated in the preceding article. If, however, we suppose m large, and are content to retain only the most important term in Equation 2.64, we simply get

$$\Delta k = \frac{3a^3}{2 \cdot (b^3 - a^3)}\,, \quad \Delta k' = 0\,, \qquad (2.65)$$

so that the correction for the envelope may be calculated as if the fluid were destitute of friction.

2.3. Solution of the equations in the case of an infinite cylinder oscillating in an unlimited mass of fluid, in a direction perpendicular to its axis.

24. Suppose a long cylindrical rod suspended at a point in its axis, and made to oscillate as a pendulum in an unlimited mass of fluid. The resistance experienced by any element of the cylinder comprised between two parallel planes drawn perpendicular to the axis will manifestly be very nearly the same as if the element belonged to an infinite cylinder oscillating with the same linear velocity. For an element situated very near either extremity of the rod, the resistance thus determined would, no doubt, be sensibly erroneous; but as the diameter of the rod is supposed to be but small in comparison with its length, it will be easily seen that the error thus introduced must be extremely small.
Imagine then an infinite cylinder to oscillate in a fluid, in a direction perpendicular to its axis, so that the motion takes place in two dimensions, and let it be required to determine the motion of the fluid. The mode of solution of this problem will require no explanation, being identical in principle with that which has been already adopted in the case of a sphere. In the present instance, the problem will be found somewhat easier, up to the formation of the equations analogous to Equation 2.33 and Equation 2.34, after which it will become much more difficult.

25. Let a plane drawn perpendicular to the axis of the cylinder be taken for the plane of xy, the origin being situated in the mean position of the axis of the cylinder, and the axis of x being measured in the direction of the cylinder's motion.

The general Equation 2.2 and Equation 2.3 become in this case

$$\left.\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}x} &= \mu \cdot \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right) - \rho \cdot \frac{\mathrm{d}u}{\mathrm{d}t}\,, \\ \frac{\mathrm{d}p}{\mathrm{d}y} &= \mu \cdot \left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \frac{\mathrm{d}^2v}{\mathrm{d}y^2}\right) - \rho \cdot \frac{\mathrm{d}v}{\mathrm{d}t}\,, \end{aligned}\right\} \tag{2.66}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} = 0\,. \tag{2.67}$$

By virtue of Equation 2.67, $u\,\mathrm{d}y - v\,\mathrm{d}x$ is an exact differential. Let then

$$u\,\mathrm{d}y - v\,\mathrm{d}x = \mathrm{d}\chi\,. \tag{2.68}$$

Eliminating p by differentiation from the two Equation 2.66, and expressing u and v in terms of χ in the resulting equation, we get

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{1}{\mu'} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\right) \cdot \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2}\right) \chi = 0\,, \tag{2.69}$$

and, as before

$$\chi = \chi_1 + \chi_2 \tag{2.70}$$

where

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2}\right) \chi_1 = 0\,, \tag{2.71}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{1}{\mu'} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\right) \chi_2 = 0\,. \tag{2.72}$$

We get from Equation 2.66 and Equation 2.68

$$\mathrm{d}p = \mu'\rho\,\mathrm{d}x \cdot \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{1}{\mu'} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\right) \chi - \mu'\rho\,\mathrm{d}y \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{1}{\mu'} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\right) \chi\,,$$

which becomes by means of Equation 2.70, Equation 2.71, and Equation 2.72

$$\mathrm{d}p = \rho \cdot \left(\frac{\mathrm{d}\chi_1}{\mathrm{d}t\,\mathrm{d}x}\,\mathrm{d}y - \frac{\mathrm{d}\chi_1}{\mathrm{d}t\,\mathrm{d}y}\,\mathrm{d}x\right)\,. \tag{2.73}$$

26. Passing to polar co-ordinates r, θ, where θ is supposed to be measured from the axis of x, we get from Equation 2.68, Equation 2.71, Equation 2.72, and Equation 2.73

$$Rr\,\mathrm{d}\theta - \Theta\,\mathrm{d}r = \mathrm{d}\chi\,, \tag{2.74}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r} \cdot \frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r^2} \cdot \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\right) \chi_1 = 0\,, \tag{2.75}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\cdot\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r^2}\cdot\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}-\frac{1}{\mu'}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\right)\chi_2=0\,, \tag{2.76}$$

$$\mathrm{d}p=\rho\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\chi_1}{\mathrm{d}r}\cdot r\,\mathrm{d}\theta-\frac{1}{r}\cdot\frac{\mathrm{d}\chi_1}{\mathrm{d}\theta}\,\mathrm{d}r\right)\,, \tag{2.77}$$

R and Θ in Equation 2.74 being the velocities along and perpendicular to the radius vector.

27. Let a be the radius of the cylinder; and as before let the cylinder's motion be defined by the equation

$$\frac{\mathrm{d}\xi}{\mathrm{d}t}=c\cdot e^{\mathbf{i}\,nt}=c\cdot e^{\mu' m^2 t}\,; \tag{2.78}$$

then we have for the equations of condition which relate to the surface of the cylinder

$$\left.\begin{aligned} R&=\frac{1}{r}\cdot\frac{\mathrm{d}\chi}{\mathrm{d}\theta}=\cos(\theta)\cdot\frac{\mathrm{d}\xi}{\mathrm{d}t}=c\cdot\cos(\theta)\cdot e^{\mu' m^2 t}\,,\\ \Theta&=-\frac{\mathrm{d}\chi}{\mathrm{d}r}=-\sin(\theta)\cdot\frac{\mathrm{d}\xi}{\mathrm{d}t}=-c\cdot\sin(\theta)\cdot e^{\mu' m^2 t}\,, \end{aligned}\right\} \tag{2.79}$$

when $r=a$.
The general Equation 2.75, Equation 2.76, as well as the equations of condition, Equation 2.79, may be satisfied by taking

$$\left.\begin{aligned} \chi_1&=e^{\mu' m^2 t}\cdot\sin(\theta)\cdot F_1(r)\,,\\ \chi_2&=e^{\mu' m^2 t}\cdot\sin(\theta)\cdot F_2(r)\,. \end{aligned}\right\} \tag{2.80}$$

Substituting in Equation 2.75, Equation 2.76, and Equation 2.79, we get

$$F_1''(r)+\frac{1}{r}\cdot F_1'(r)-\frac{1}{r^2}\cdot F_1(r)=0\,, \tag{2.81}$$

$$F_2''(r)+\frac{1}{r}\cdot F_2'(r)-\frac{1}{r^2}\cdot F_2(r)-m^2\cdot F_2(r)=0\,, \tag{2.82}$$

$$\left.\begin{aligned} F_1(a)+F_2(a)&=a\cdot c\,,\\ F_1'(a)+F_2'(a)&=c\,, \end{aligned}\right\} \tag{2.83}$$

besides which we have the condition that the velocity shall vanish at an infinite distance.

28. The integral of Equation 2.81 is

$$F_1(r) = \frac{A}{r} + Br\,. \tag{2.84}$$

The integral of Equation 2.82 cannot be obtained in finite terms.

To simplify the latter equation, assume that $F_2(r) = F_3'(r)$. Substituting in Equation 2.82, and integrating once, we get

$$F_3''(r) + \frac{1}{r}\cdot F_3'(r) - m^2\cdot F_3(r) = 0\,. \tag{2.85}$$

It is unnecessary to add an arbitrary constant, because such a constant, if introduced, might be got rid of by writing $F_3(r) + C$ for $F_3(r)$.
To integrate Equation 2.85 by series according to ascending powers of r, let us first, instead of Equation 2.85, take the equation formed from it by multiplying the second term by $(1-\delta)$. Assuming in this new equation $F_3(r) = A_\prime x^\alpha + B_\prime x^\beta + \cdots$, and determining the arbitrary indices α, β, ... and the arbitrary constants $A_\prime$, $B_\prime$, ... so as to satisfy the equation, we get

$$\begin{aligned}
F_3(r) &= A_\prime\cdot\left\{1 + \frac{m^2r^2}{2\cdot(2-\delta)} + \frac{m^4r^4}{2\cdot4\cdot(2-\delta)\cdot(4-\delta)} + \cdots\right\}\\
&\quad + A_{\prime\prime}r^\delta\cdot\left\{1 + \frac{m^2r^2}{2\cdot(2+\delta)} + \frac{m^4r^4}{2\cdot4\cdot(2+\delta)\cdot(4+\delta)} + \cdots\right\}\\
&= (A_\prime + A_{\prime\prime} + A_{\prime\prime}\cdot\delta\log[r])\cdot\left\{1 + \frac{m^2r^2}{2^2} + \frac{m^4r^4}{2^2\cdot4^2} + \cdots\right\}\\
&\quad + \frac{1}{2}\cdot(A_\prime - A_{\prime\prime})\delta\cdot\left\{\frac{m^2r^2}{2^2}S_1 + \frac{m^4r^4}{2^2\cdot4^2}S_2 + \frac{m^6r^6}{2^2\cdot4^2\cdot6^2}S_3 + \cdots\right\}\\
&\quad + \text{ terms involving } \delta^2, \delta^3, \ldots
\end{aligned}$$

In this expression

$$S_i = 1^{-1} + 2^{-1} + 3^{-1} + \ldots + i^{-1}\,. \tag{2.86}$$

Putting now

$$A_\prime = C - A_{\prime\prime}\,, \qquad A_{\prime\prime} = D\delta^{-1}\,,$$

and substituting in the above equation, and then making δ vanish, we get

$$\begin{aligned}
F_3(r) &= [C + D\log(r)]\cdot\left(1 + \frac{m^2r^2}{2^2} + \frac{m^4r^4}{2^2\cdot4^2} + \cdots\right)\\
&\quad - D\cdot\left(\frac{m^2r^2}{2^2}S_1 + \frac{m^4r^4}{2^2\cdot4^2}S_2 + \frac{m^6r^6}{2^2\cdot4^2\cdot6^2}S_3 + \cdots\right)\,.
\end{aligned} \tag{2.87}$$

The series in this equation are evidently convergent for all values of r, however great;

but, nevertheless, they give us no information as to what becomes of $F_3(r)$ when r becomes infinite, and yet one relation between C and D has to be determined by the condition that $F_3(r)$ shall not become infinite with r.
Equation 2.85 may be integrated by means of descending series combined with exponentials, by assuming $F_3(r) = e^{\pm mr} \cdot \left(A_{\prime} r^{\alpha} + B_{\prime} r^{\beta} + \cdots\right)$. I have already given the integral in this form in a paper, *On the numerical calculation of a class of definite integrals and infinite series*[8]. The result is

$$\begin{aligned} F_3(r) = C' \cdot e^{-mr} \frac{1}{\sqrt{r}} \cdot \Bigg\{ 1 - \frac{1^2}{2 \cdot 4mr} + \frac{1^2 \cdot 3^2}{2 \cdot 4 \cdot (4mr)^2} \\ - \frac{1^2 \cdot 3^2 \cdot 5^2}{2 \cdot 4 \cdot 6 \cdot (4mr)^3} + \cdots \Bigg\} \\ + D' \cdot e^{mr} \frac{1}{\sqrt{r}} \cdot \Bigg\{ 1 + \frac{1^2}{2 \cdot 4mr} + \frac{1^2 \cdot 3^2}{2 \cdot 4 \cdot (4mr)^2} \\ + \frac{1^2 \cdot 3^2 \cdot 5^2}{2 \cdot 4 \cdot 6 \cdot (4mr)^3} + \cdots \Bigg\} . \end{aligned} \tag{2.88}$$

These series, although ultimately divergent in all cases, are very convenient for numerical calculation when the modulus of mr is large. Moreover they give at once $D' = 0$ for the condition that $F_3(r)$ shall not become infinite with r, and therefore we shall be able to obtain the required relation between C and D, provided we can express D' as a function of C and D.

29. This may be effected by means of the integral of Equation 2.85 expressed by definite integrals. This form of the integral is already known. It becomes, by a slight transformation,

$$F_3(r) = \int_0^{\pi/2} \left\{ C'' + D'' \log \left[r \cdot \sin^2(\omega) \right] \right\} \cdot \left(e^{mr \cdot \cos(\omega)} + e^{-mr \cdot \cos(\omega)} \right) \mathrm{d}\omega \,, \tag{2.89}$$

C'' and D'' being the two arbitrary constants. If we expand the exponentials in Equation 2.89, and integrate the terms separately, we obtain, in fact, an expression of the same form as Equation 2.87. This transformation requires the reduction of the definite integral

$$P_i = \int_0^{\pi/2} \cos^{2i}(\omega) \cdot \log \left[\sin(\omega) \right] \mathrm{d}\omega \,.$$

If we integrate by parts, integrating $\cos(\omega) \cdot \log \left[\sin(\omega) \right] \mathrm{d}\omega$, and differentiating $\cos^{2i-1} \cdot \omega$, we shall make P_i depend on P_{i-1}. Assuming $P_0 = Q_0$, $P_1 = \frac{1}{2} Q_1$, ... and generally

$$P_i = \frac{1 \cdot 3 \cdot \ldots \cdot (2i-1)}{2 \cdot 4 \cdot \ldots \cdot 2i} \cdot Q_i \,,$$

[8] *Camb. Phil. Trans.* Vol. IX. p. 182.

we get[9]

$$Q_i = Q_0 - \left\{2^{-1} + 4^{-1} + \cdots + (2i)^{-1}\right\} \cdot \frac{\pi}{2} = \frac{\pi}{2} \cdot \log\left(\frac{1}{2}\right) - \frac{\pi}{4} \cdot S_i.$$

The equivalence of Equation 2.87 and Equation 2.89 having been ascertained, in order to find the relations between C, D and C'', D'', it will be sufficient to write down the two leading terms in Equation 2.87 and Equation 2.89, and equate the results. We thus get

$$C + D \cdot \log(r) = \pi C'' + \pi D'' \cdot \log(r) + 2\pi D'' \cdot \log\left(\frac{1}{2}\right),$$

whence

$$C = \pi C'' + 2\pi D'' \cdot \log\left(\frac{1}{2}\right), \quad D = \pi D''. \tag{2.90}$$

There remains the more difficult step of finding the relation between D' and C'', D''. For this purpose let us seek the ultimate value of the second member of Equation 2.89 when r increases indefinitely. In the first place we may observe that if Ω, Ω' be two imaginary quantities having their real parts positive, if the real part of Ω be greater than that of Ω', and if r be supposed to increase indefinitely, $e^{\Omega r}$ will ultimately be incomparably greater than $e^{\Omega' r}$, or even than $\log(r) \cdot e^{\Omega' r}$, or, to speak more precisely, the modulus of the former expression will ultimately be incomparably greater than the modulus of either of the latter. Hence, in finding the ultimate value of the expression for $F_3(r)$ in Equation 2.89, we may replace the limits 0 and $\frac{1}{2}\pi$ of ω by 0 and ω_1, where ω_1 is a positive quantity as small as we please, which we may suppose to vanish after r has become infinite. We may also, for the same reason, omit the second of the exponentials. Let $\cos(\omega) = (1 - \lambda)$, so that

$$\sin^2(\omega) = 2\lambda \cdot \left(1 - \frac{\lambda}{2}\right),$$
$$\mathrm{d}\omega = \frac{\mathrm{d}\lambda}{\sqrt{2\lambda - \lambda^2}} = \left(1 + \frac{\lambda}{4} + \cdots\right) \cdot \frac{\mathrm{d}\lambda}{\sqrt{2\lambda}};$$

then the limits of λ will be 0 and λ_1, where $\lambda_1 = [1 - \cos(\omega_1)]$. Since $\log\left(1 - \frac{\lambda}{2}\right)$ ultimately vanishes, and $1 + \frac{\lambda}{4} + \cdots$ becomes ultimately 1, we get from Equa-

[9] A demonstration by Mr Ellis of the theorem

$$\int_0^{\pi/2} \log\left[\sin(\theta)\right] \mathrm{d}\theta = \frac{\pi}{2} \cdot \log\left(\frac{1}{2}\right)$$

due to Euler will be found in the 2nd volume of the *Cambridge Mathematical Journal*, p. 282, or in Gregory's Examples, p. 484.

tion 2.89[10]

$$\text{limit of } F_s(r) = e^{mr} \times \text{ limit of } \int_0^{\lambda_1} [C'' + D'' \log(2\lambda r)] \cdot e^{-m\lambda r} \cdot \frac{\mathrm{d}\lambda}{\sqrt{2\lambda}}\,.$$

If now we put $\lambda = \frac{\lambda'}{r}$, we shall have 0 and $\lambda_1 \cdot r$ for the limits of λ', and the second of these becomes infinite with r. Hence

$$\text{limit of } F_3(r) = \frac{1}{\sqrt{2r}} \cdot e^{mr} \cdot \int_0^{\infty} [C'' + D'' \log(2\lambda')] \cdot e^{-m\lambda'} \frac{\mathrm{d}\lambda'}{\sqrt{\lambda'}}\,. \tag{2.91}$$

Now

$$\int_0^{\infty} \frac{1}{e^x \cdot \sqrt{x}}\,\mathrm{d}x = \sqrt{\pi}\,,$$

and if we differentiate both sides of the equation[11]

$$\int_0^{\infty} \frac{x^{s-1}}{e^x}\,\mathrm{d}x = \Gamma(s) \quad \text{with} \quad \mathrm{Re}(s) > 0$$

with respect to s, and after differentiation put $s = \frac{1}{2}$, we get

$$\int_0^{\infty} \frac{\log(x)}{e^x \cdot \sqrt{x}}\,\mathrm{d}x = \Gamma'\left(\frac{1}{2}\right)\,.$$

Putting $x = m \cdot \lambda'$ in these equations we get

$$\int_0^{\infty} e^{-m\lambda'} \cdot \frac{\mathrm{d}\lambda'}{\sqrt{\lambda'}} = \sqrt{\frac{\pi}{m}}\,,$$
$$\int_0^{\infty} e^{-m\lambda'} \cdot \frac{\log(\lambda')\,\mathrm{d}\lambda'}{\sqrt{\lambda'}} = \frac{1}{\sqrt{m}} \cdot \left\{\Gamma'\left(\frac{1}{2}\right) - \sqrt{\pi} \cdot \log(m)\right\}\,,$$

where that value of $\frac{1}{\sqrt{m}}$ is to be taken which has its real part positive. Substituting

[10] The word *limit* is here used in the sense in which $f(r)$ may be called the limit of $\phi(r)$ when the ratio of $\phi(r)$ to $f(r)$ is ultimately a ratio of equality, though $f(r)$ and $\phi(r)$ may vanish or become infinite together, in which case the limit of $\phi(r)$, according to the usual sense of the word *limit*, would be said to be zero or infinity.

[11] This is one possible definition of the Gamma function. In general, there is no uniform definition for the gamma function in the literature. EULER's integral of the second kind is often given. A disadvantage is that this integral does not converge everywhere. A global calculation using this definition is therefore only possible indirectly. For complex numbers z with a positive real part, the gamma function is therefore the improper integral

$$\Gamma(z) = \int_0^{\infty} t^{z-1} \mathrm{e}^{-t}\,\mathrm{d}t\,.$$

in Equation 2.91 we get

$$\text{limit of } F_3(r) = \sqrt{\frac{\pi}{2mr}} \cdot e^{mr} \cdot \left\{ C'' + D'' \cdot \left[\frac{1}{\sqrt{\pi}} \cdot \Gamma'\left(\frac{1}{2}\right) - \log\left(\frac{m}{2}\right) \right] \right\} .$$

Comparing with Equation 2.88 we get

$$D' = \sqrt{\frac{\pi}{2m}} \cdot \left\{ C'' + D'' \cdot \left[\frac{1}{\sqrt{\pi}} \cdot \Gamma'\left(\frac{1}{2}\right) - \log\left(\frac{m}{2}\right) \right] \right\} . \tag{2.92}$$

30. We are now enabled to find the relation between C and D arising from the condition that the motion of the fluid shall not become infinitely great at an infinite distance from the cylinder. The determination of the arbitrary constants A, B, C, D will present no further difficulty. We must have $B = 0$, since otherwise the velocity would be finite at an infinite distance, and then the two of Equation 2.83, combined with the relation above mentioned, will serve to determine A, C, D. The motion of the fluid will thus be completely determined, the functions $F_1(r)$, $F_3(r)$ being given by Equation 2.84 and Equation 2.87. When the modulus of mr is large, the series in Equation 2.87, though ultimately hypergeometrically convergent, are at first rapidly divergent, and in calculating the numerical value of $F_3(r)$ in such a case it would be far more convenient to employ Equation 2.88. The employment of this equation for the purpose would require the previous determination of the constant C'. It will be found however that in calculating the resultant pressure of the fluid on the cylinder, which it is the main object of the present investigation to determine, a knowledge of the value of C' will not be required, and that, even though the Equation 2.88 be employed.
Putting $D' = 0$ in Equation 2.92, and eliminating C'' and D'' between the resulting equation and the two of Equation 2.90, we get

$$C = D \cdot \left[\log\left(\frac{m}{8}\right) - \frac{1}{\sqrt{\pi}} \cdot \Gamma'\left(\frac{1}{2}\right) \right] ; \tag{2.93}$$

and we get from Equation 2.83 and Equation 2.84, observing that $F_2(r) = F_3'(r)$, and that $B = 0$,

$$\left.\begin{aligned} \frac{A}{a} + F_3'(a) &= a \cdot c\,, \\ -\frac{A}{a} + a \cdot F_3''(a) &= a \cdot c\,, \end{aligned}\right\} \tag{2.94}$$

whence

$$\frac{a^2c + A}{a^2c - A} = \frac{a \cdot F_3''(a)}{F_3'(a)} . \tag{2.95}$$

This equation will determine A, because if $F_3(a)$ be expressed by Equation 2.87,

the second member of Equation 2.95 will only contain the ratio of C to D, which is given by Equation 2.93, and if $F_3(a)$ be expressed by Equation 2.88, C' will disappear, inasmuch as $D' = 0$.

31. Let us now form the expression for the resultant of the forces which the fluid exerts on the cylinder. Let $\mathscr{F}$ be the resultant of the pressures acting on a length $\mathrm{d}l$ of the cylinder, which will evidently be a force acting in the direction of the axis of x; then we get in the same way as the Equation 2.47 was obtained

$$\mathscr{F} = a\,\mathrm{d}l \cdot \int_0^{2\pi} \left[-P_r \cdot \cos(\theta) + T_\theta \cdot \sin(\theta)\right]_a \mathrm{d}\theta\,, \tag{2.96}$$

and P_r and T_θ are given in terms of R and Θ by the same formulæ, Equation 2.46, as before. When the right-hand members of these equations are expressed in terms of χ, there will be only one term in which the differentiation with respect to r rises to the second order, and we get from Equation 2.70, Equation 2.75, and Equation 2.76

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}r^2} = -\frac{1}{r}\cdot\frac{\mathrm{d}\chi}{\mathrm{d}r} - \frac{1}{r^2}\cdot\frac{\mathrm{d}^2\chi}{\mathrm{d}\theta^2} + \frac{1}{\mu'}\cdot\frac{\mathrm{d}\chi_2}{\mathrm{d}t}\,.$$

We get from this equation and the equations of condition, Equation 2.79

$$\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)_a = \frac{1}{a}\cdot\left(\frac{\mathrm{d}\chi}{\mathrm{d}\theta}\right)_a - \frac{1}{a^2}\cdot\left(\frac{\mathrm{d}\chi}{\mathrm{d}r\,\mathrm{d}\theta}\right)_a = 0\,,$$

$$\left(\frac{1}{r}\cdot\frac{\mathrm{d}R}{\mathrm{d}\theta}\right)_a = \frac{1}{a^2}\cdot\left(\frac{\mathrm{d}^2\chi}{\mathrm{d}\theta^2}\right)_a = -\frac{\sin(\theta)}{a}\cdot\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\Theta}{a}\,,$$

$$\left(\frac{\mathrm{d}\Theta}{\mathrm{d}r}\right)_a = -\left(\frac{\mathrm{d}^2\chi}{\mathrm{d}r^2}\right)_a = \frac{1}{a}\cdot\left(\frac{\mathrm{d}\chi}{\mathrm{d}r}\right)_a + \frac{1}{a^2}\cdot\left(\frac{\mathrm{d}^2\chi}{\mathrm{d}\theta^2}\right)_a - \frac{1}{\mu'}\cdot\left(\frac{\mathrm{d}\chi_2}{\mathrm{d}t}\right)_a = -\frac{1}{\mu'}\cdot\left(\frac{\mathrm{d}\chi_2}{\mathrm{d}t}\right)_a\,.$$

Hence

$$\mathscr{F} = a\,\mathrm{d}l \cdot \int_0^{2\pi} \left\{-p_a \cdot \cos(\theta) + \rho \cdot \left(\frac{\mathrm{d}\chi_2}{\mathrm{d}t}\right)_a \cdot \sin(\theta)\right\} \mathrm{d}\theta\,. \tag{2.97}$$

We get by integration by parts

$$\int p_a \cdot \cos(\theta)\,\mathrm{d}\theta = p_a \cdot \sin(\theta) - \int \left(\frac{\mathrm{d}p}{\mathrm{d}\theta}\right)_a \cdot \sin(\theta)\,\mathrm{d}\theta\,.$$

The first term vanishes at both limits; and putting for $\frac{\mathrm{d}p}{\mathrm{d}\theta}$ its value given by Equation 2.77, and substituting in Equation 2.97, we get

$$\mathscr{F} = \rho a\,\mathrm{d}l \cdot \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} \left\{a \cdot \left(\frac{\mathrm{d}\chi_1}{\mathrm{d}r}\right)_a + (\chi_2)_a\right\} \cdot \sin(\theta)\,\mathrm{d}\theta\,,$$

or

$$\mathscr{F} = \pi\rho a\,\mathrm{d}l \cdot n\,\mathbf{i} \cdot \{a \cdot F_1'(a) + F_3'(a)\} \cdot e^{\mathbf{i}\,nt}\,.$$

Observing that $F_3'(a)$ or $F_2(a) = ac - F_1(a)$ from Equation 2.83, and that $F_1(a) = \frac{A}{a}$, where A is given by Equation 2.95, and putting M' for $\pi\rho a^2\,\mathrm{d}l$, the mass of the fluid displaced, we get

$$\mathscr{F} = M'cn\,\mathbf{i} \cdot \left\{1 - 2 \cdot \frac{aF_3''(a) - F_3'(a)}{aF_3''(a) + F_3'(a)}\right\} \cdot e^{\mathbf{i}\,nt}\,,$$

which becomes by means of the differential Equation 2.85 which F_3 satisfies

$$\mathscr{F} = -M'cn\,\mathbf{i} \cdot \left\{1 - \frac{4 \cdot F_3'(a)}{m^2 a \cdot F_3(a)}\right\} \cdot e^{\mathbf{i}\,nt}\,. \tag{2.98}$$

Let

$$1 - \frac{4 \cdot F_3'(a)}{m^2 a \cdot F_3(a)} = k - \mathbf{i}\,k'\,, \tag{2.99}$$

where k and k' are real, then, as before, $kM' \cdot \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}$ will be the part of $\mathscr{F}$ which alters the time of oscillation, and $k'M'n \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t}$ the part which produces a diminution in the arc of oscillation.
When μ' vanishes, m becomes infinite, and we get from Equation 2.88 and Equation 2.99, remembering that $D' = 0$; $k = 1$, $k' = 0$, a result which follows directly and very simply from the ordinary equations of hydrodynamics[12].

32. Everything is now reduced to the numerical calculation of the quantities k, k', of which the analytical expressions are given. The series in Equation 2.87 being always convergent might be employed in all cases, but when the modulus of ma is large, it will be far more convenient to employ a series according to descending powers of a. Let us consider the ascending series first. Let $2\,\mathfrak{m}$ be the modulus of ma; then

$$\left.\begin{aligned} ma &= 2\,\mathfrak{m} \cdot e^{\frac{\pi}{4}\mathbf{i}}\,, \\ \mathfrak{m} &= \frac{a}{2} \cdot \sqrt{\frac{n}{\mu'}} = \frac{a}{2} \cdot \sqrt{\frac{\pi}{\mu' \cdot \tau}}\,, \end{aligned}\right\} \tag{2.100}$$

τ being as before the time of oscillation from rest to rest. Substituting in Equation 2.99 the above expression for ma, we get

$$k - \mathbf{i}\,k' = 1 + \frac{\mathbf{i}\,a \cdot F_3'(a)}{\mathfrak{m}^2\,F_3(a)} \tag{2.101}$$

[12]See *Camb. Phil. Trans.* Vol. VIII. p. 35.

Putting for shortness

$$\ln(4) + \frac{1}{\sqrt{\pi}} \cdot \Gamma'\left(\frac{1}{2}\right) = -\Lambda \tag{2.102}$$

we get from Equation 2.87 and Equation 2.93

$$\begin{aligned}\frac{1}{D} \cdot F_3(a) = &\left[\log(\mathfrak{m}) + \Lambda + \frac{\pi}{4}\,\mathbf{i}\right] \cdot \left[1 + \frac{\mathfrak{m}^2}{1^2}\,\mathbf{i} - \frac{\mathfrak{m}^4}{1^2 \cdot 2^2} - \frac{\mathfrak{m}^6}{1^2 \cdot 2^2 \cdot 3^2}\,\mathbf{i} + \cdots\right] \\ &- \left[\frac{\mathfrak{m}^2}{1^2} S_1\,\mathbf{i} - \frac{\mathfrak{m}^4}{1^2 \cdot 2^2} S_2 - \frac{\mathfrak{m}^6}{1^2 \cdot 2^2 \cdot 3^2} S_3\,\mathbf{i} + \cdots\right],\end{aligned}$$

$$\begin{aligned}\frac{1}{D} a \cdot F_3'(a) = &1 + \frac{\mathfrak{m}^2}{1^2}\,\mathbf{i} - \frac{\mathfrak{m}^4}{1^2 \cdot 2^2} - \cdots + 2 \cdot \left[\log(\mathfrak{m}) + \Lambda + \frac{\pi}{4}\,\mathbf{i}\right] \cdot \left(\frac{\mathfrak{m}^2}{1}\,\mathbf{i} - \frac{\mathfrak{m}^4}{1^2 \cdot 2} - \cdots\right) \\ &- 2 \cdot \left[\frac{\mathfrak{m}^2}{1} S_1\,\mathbf{i} - \frac{\mathfrak{m}^4}{1^2 \cdot 2} S_2 - \frac{\mathfrak{m}^6}{1^2 \cdot 2^2 \cdot 3} S_3\,\mathbf{i} + \cdots\right].\end{aligned}$$

Let

$$\left.\begin{aligned}
&\frac{\mathfrak{m}^2}{1} - \frac{\mathfrak{m}^6}{1^2 \cdot 2^2 \cdot 3} + \cdots = M_0\,, \\
&\frac{\mathfrak{m}^2}{1^2} - \frac{\mathfrak{m}^6}{1^2 \cdot 2^2 \cdot 3^2} + \cdots = M_0'\,, \\
&\frac{\mathfrak{m}^4}{1^2 \cdot 2} - \frac{\mathfrak{m}^8}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4} + \cdots = M_e\,, \\
&\frac{\mathfrak{m}^4}{1^2 \cdot 2^2} - \frac{\mathfrak{m}^8}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2} + \cdots = M_e'\,, \\
&\frac{\mathfrak{m}^2}{1} S_1 - \frac{\mathfrak{m}^6}{1^2 \cdot 2^2 \cdot 3} S_2 + \cdots = N_0\,, \\
&\frac{\mathfrak{m}^2}{1^2} S_1 - \frac{\mathfrak{m}^6}{1^2 \cdot 2^2 \cdot 3^2} S_3 + \cdots = N_0'\,, \\
&\frac{\mathfrak{m}^4}{1^2 \cdot 2} S_2 - \frac{\mathfrak{m}^8}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4} S_4 + \cdots = N_e\,, \\
&\frac{\mathfrak{m}^4}{1^2 \cdot 2^2} S_2 - \frac{\mathfrak{m}^8}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2} S_4 + \cdots = N_e'\,,
\end{aligned}\right\} \tag{2.103}$$

$$\ln(\mathfrak{m}) + \Lambda = L\ : \tag{2.104}$$

then substituting in Equation 2.101, changing the sign of $\mathbf{i}$, and arranging the

terms, we get

$$k + \mathbf{i}\,k' = 1 + \frac{2}{\mathfrak{m}^2} \cdot \frac{-LM_0 + \frac{\pi}{4}M_e - \frac{1}{2}M_0' + N_0 + \left\{\frac{\pi}{4}M_0 + LM_e - \frac{1}{2}\cdot(1 - M_e') - N_e\right\}\mathbf{i}}{-\frac{\pi}{4}M_0' + L\cdot(1 - M_e') + N_e' + \left\{-LM_0' - \frac{\pi}{4}\cdot(1 - M_e') + N_0'\right\}\mathbf{i}}\,. \quad (2.105)$$

33. Before going on with the calculation, it will be requisite to know the numerical value of the transcendental quantity Λ. Now

$$\frac{1}{\sqrt{\pi}}\cdot\Gamma'\left(\frac{1}{2}\right) = \frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\mathrm{d}}{\mathrm{d}s}\log\Gamma(s) = \frac{\mathrm{d}}{\mathrm{d}s}\log\left[\Gamma(1+s)\right] - \frac{1}{s}\,, \quad \text{for} \quad s = \frac{1}{2}$$

and the value of $\frac{\mathrm{d}}{\mathrm{d}s}\log\left[\Gamma(1+s)\right]$ may be got at once from LEGENDRE's table of the common logarithms[13] of $\Gamma(1+s)$, in which the interval of s is 0.001. Putting l_s for the tabular number corresponding to s, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\log\left[\Gamma(1+s)\right] = 1000\cdot\ln(10)\cdot\left\{\Delta l_s - \frac{1}{2}\cdot\Delta^2 l_s + \frac{1}{3}\cdot\Delta^3 l_s - \frac{1}{4}\cdot\Delta^4 l_s + \cdots\right\}.$$

For $s = \frac{1}{2}$ we get[14]

$$\Delta l_s = +16050324\,, \quad \Delta^2 l_s = +405620\,, \quad \Delta^3 l_s = -359\,, \quad \Delta^4 l_s = +6\,,$$

the last figure being in each case in the 12th place of decimals. We thus get

$$\frac{1}{\sqrt{\pi}}\cdot\Gamma'\left(\frac{1}{2}\right) = -1.9635102\,, \quad \Lambda = +0.5772158\,. \quad (2.106)$$

34. When $\mathfrak{m}$ is large, it will be more convenient to employ series according to descending powers of a. Observing that the general term of $F_3(a)$ as given by Equation 2.88, in which $D' = 0$, is

$$(-1)^i C'\cdot e^{-ma}\cdot\frac{\left[1\cdot 3\cdot\ldots\cdot(2i-1)\right]^2}{2\cdot 4\cdot\ldots\cdot 2i\cdot(4ma)^i\cdot\sqrt{a}}\,,$$

we get for the general term of $F_3'(a)$

$$(-1)^{i-1}C'\cdot e^{-ma}\cdot\frac{\left[1\cdot 3\cdot\ldots\cdot(2i-1)\right]^2}{2\cdot 4\cdot\ldots\cdot(2i-2)\cdot(4ma)^{i-1}\cdot\sqrt{a}}\cdot\left\{m\cdot\frac{(2i-1)^2}{2i\cdot 4ma} - \frac{2i-1}{2a}\right\},$$

[13] https://books.google.de/books?id=c4M_AAAAcAAJ&hl=de&source=gbs_navlinks_s

[14] The following numbers are copied from DE MORGAN's *Differential and Integral Calculus*, p. 588.

and the expression within brackets is equivalent to

$$-\frac{(2i-1)\cdot(2i+1)}{8ia}\,,$$

whence

$$a\cdot F_3'(a) = C'\cdot e^{-ma}\cdot m\sqrt{a}\cdot\left\{-1-\frac{1\cdot 3}{2\cdot 4ma}+\frac{1^2\cdot 3\cdot 5}{2\cdot 4\cdot(4ma)^2}-\cdots\right\},$$

and we find by actual division

$$\frac{a\cdot F_3'(a)}{F_3(a)} = -ma-\frac{1}{2}+\frac{1}{8}\cdot\frac{1}{(ma)}\cdots$$

35. When many terms are required, the calculation of the coefficients may be facilitated in the following manner.

Assuming $aF_3'(a) = v(a)F_3(a)$, we have

$$F_3'(a) = \frac{v(a)}{a}\cdot F_3(a)\,,$$
$$F_3''(a) = \left\{\frac{v'(a)}{a}-\frac{v(a)}{a^2}+\frac{(va)^2}{a^2}\right\}\cdot F_3(a)\,.$$

Substituting in the differential Equation 2.85 which F_3 has to satisfy, we get

$$a\cdot v'(a)+[v(a)]^2-m^2a^2 = 0\,. \tag{2.107}$$

Assuming

$$v(a) = -ma+A_0+\frac{A_1}{ma}+\frac{A_2}{(ma)^2}+\cdots\,, \tag{2.108}$$

and substituting in the above equation, we get

$$-ma-\frac{1\cdot A_1}{ma}-\frac{2\cdot A_2}{(ma)^2}-\frac{3\cdot A_3}{(ma)^3}-\cdots$$
$$+\left\{-2ma+A_0+\frac{A_1}{ma}+\cdots\right\}\cdot\left\{A_0+\frac{A_1}{ma}+\cdots\right\} = 0\,,$$

which gives on equating coefficients, $A_0 = -\frac{1}{2}$, and for $i>0$

$$2\cdot A_{i+1} = -iA_i+A_0A_i+A_1A_{i-1}+\cdots+A_iA_0\,,$$

or, assuming to avoid fractions,

$$A_i = 2^{-2i-1}\cdot B_i\,, \tag{2.109}$$

$$B_{i+1} = -2iB_i + B_0B_i + B_1B_{i-1} + \cdots + B_iB_0\,, \tag{2.110}$$

a formula by means of which the coefficients B_1, B_2, B_3, ... may be readily calculated one after another. We get

$$\left.\begin{array}{ccc} B_0 = -1\,, & B_1 = +1\,, & B_2 = -4\,, \\ B_3 = +25\,, & B_4 = -208\,, & B_5 = +2146\,, \\ B_6 = -26368\,, & B_7 = +375733\,, & B_8 = -6092032\,. \end{array}\right\} \tag{2.111}$$

We get now from Equation 2.100, Equation 2.101, Equation 2.108, and Equation 2.109

$$k - \mathbf{i}\,k' = 1 + 2\cdot e^{-\frac{\pi}{4}\mathbf{i}}\cdot\frac{1}{\mathfrak{m}} - \frac{1}{2}B_0\cdot e^{-2\frac{\pi}{4}\mathbf{i}}\cdot\frac{1}{\mathfrak{m}^2} - \frac{1}{2^4}B_1\cdot e^{-3\frac{\pi}{4}\mathbf{i}}\cdot\frac{1}{\mathfrak{m}^3} - \cdots\,, \tag{2.112}$$

whence if we calculate

$$\begin{aligned} u_1 &= \frac{2}{\mathfrak{m}}\,, \\ u_2 &= -\frac{1}{2}\cdot\frac{B_0}{\mathfrak{m}^2}\,, \\ u_3 &= \frac{1}{16}\cdot\frac{B_1}{\mathfrak{m}^3}\,, \\ &\vdots \\ u_i &= (-1)^{i+1}\cdot\frac{1}{2}\cdot\frac{B_{i-2}\cdot 8^{-i+2}}{\mathfrak{m}^i}\,, \end{aligned}$$

we shall have, changing the sign of $\mathbf{i}$ in Equation 2.112, and writing $\wp$ for $e^{\frac{\pi}{4}\mathbf{i}}$,

$$\left.\begin{aligned} k + \mathbf{i}\,k' &= 1 + u_1\wp + u_2\wp^2 - u_3\wp^3 + u_4\wp^4 - u_5\wp^5 + \cdots \\ k &= 1 + \sqrt{\frac{1}{2}}u_1 + \sqrt{\frac{1}{2}}u_3 - u_4 + \sqrt{\frac{1}{2}}u_5 - \sqrt{\frac{1}{2}}u_7 + u_8 - \sqrt{\frac{1}{2}}u_9\cdots \\ k' &= \sqrt{\frac{1}{2}}u_1 + u_2 - \sqrt{\frac{1}{2}}u_3 + \sqrt{\frac{1}{2}}u_5 - u_6 + \sqrt{\frac{1}{2}}u_7 - \sqrt{\frac{1}{2}}u_9\cdots \end{aligned}\right\} \tag{2.113}$$

If l_1, l_2, ... be the common logarithms of the coefficients of $\frac{1}{\mathfrak{m}}$, $\frac{1}{\mathfrak{m}^2}$, ... in the last two of the formulæ according to Equation 2.113,

$$\begin{array}{lll} l_1 = 0.1505150\,; & l_4 = \overline{2}.4948500\,; & l_7 = \overline{2}.3646348\,; \\ l_2 = \overline{1}.6989700\,; & l_5 = \overline{2}.2371251\,; & l_8 = \overline{2}.7019316\,; \\ l_3 = \overline{2}.6453650\,; & l_6 = \overline{2}.4046734\,; & l_9 = \overline{2}.6017045\,; \end{array}$$

and if the logarithms of the coefficients of $\frac{1}{\mathfrak{m}}$, $\frac{1}{\mathfrak{m}^2}$, ... in u_1, u_2, ... be required, it will be sufficient to add 0.1505150 to the 1st, 3rd, 5th, &c. of the logarithms above given.

36. It will be found that when $\mathfrak{m}$ is at all large, the series according to Equation 2.113 are at first convergent, and afterwards divergent, and in passing from convergent to divergent the quantities u_i become nearly equal for several successive terms. If after having calculated i terms of the first of the series according to Equation 2.113 we wish to complete the series by a formula involving the differences of u_i, we have

$$\begin{aligned} u_i\wp^i - u_{i+1}\wp^{i+1} + u_{i+2}\wp^{i+2} - \cdots &= \wp^i \cdot \left\{1 - \wp\cdot(1+\Delta) + \wp^2\cdot(1+\Delta)^2 - \cdots\right\}\cdot u_i \\ &= \wp^i \cdot \frac{u_i}{\{1+\wp\cdot(1+\Delta)\}} \\ &= \frac{\wp^i}{1+\wp}\cdot\left\{1 - \frac{\wp}{1+\wp}\cdot\Delta + \left(\frac{\wp}{1+\wp}\right)^2\cdot\Delta^2 - \cdots\right\}\cdot u_i \end{aligned}$$

and

$$\begin{aligned} 1+\wp &= 1+\cos\left(\frac{\pi}{4}\right) + \mathbf{i}\sin\left(\frac{\pi}{4}\right) = 2\cdot\cos\left(\frac{\pi}{8}\right)\cdot e^{\frac{\pi}{8}\mathbf{i}}, \\ \frac{\wp}{(1+\wp)} &= \frac{1}{2}\cdot\sec\left(\frac{\pi}{8}\right)\cdot e^{\frac{\pi}{8}\mathbf{i}}, \end{aligned}$$

so that the quantities to be added to k, k', are

$$\left.\begin{aligned} \text{to } k, \quad &= (-1)^i\cdot\frac{1}{2}\cdot\sec\left(\frac{\pi}{8}\right)\cdot\left\{\cos\left(\frac{2i-1}{8}\right)\pi\cdot u_i\right. \\ &\quad -\frac{1}{2}\sec\left(\frac{\pi}{8}\right)\cos\left(\frac{2i}{8}\right)\pi\cdot\Delta u_i \\ &\quad \left.+\left[\frac{1}{2}\cdot\sec\left(\frac{\pi}{8}\right)\right]^2\cdot\cos\left(\frac{2i+1}{8}\right)\pi\cdot\Delta^2 u_i\cdots\right\} \\ \text{to } k', \quad &= (-1)^i\cdot\frac{1}{2}\cdot\sec\left(\frac{\pi}{8}\right)\cdot\left\{\sin\left(\frac{2i-1}{8}\right)\pi\cdot u_i\right. \\ &\quad -\frac{1}{2}\sec\left(\frac{\pi}{8}\right)\sin\left(\frac{2i}{8}\right)\pi\cdot\Delta u_i \\ &\quad \left.+\left[\frac{1}{2}\cdot\sec\left(\frac{\pi}{8}\right)\right]^2\cdot\sin\left(\frac{2i+1}{8}\right)\pi\cdot\Delta^2 u_i\cdots\right\} \end{aligned}\right\} \tag{2.114}$$

37. The following Table 2.1 contains the values of the functions k and k' calculated for 40 different values of $\mathfrak{m}$. From $\mathfrak{m} = 0.1$ to $\mathfrak{m} = 1.5$ the calculation was performed by means of Equation 2.105; the rest of the table was calculated by means of the series according to Equation 2.113. In the former part of the calculation, six places of decimals were employed in calculating the functions M_0,

&c. given by Equation 2.103.

$\mathfrak{m}$	k	k'	$\mathfrak{m}^2 \cdot k$	$\mathfrak{m}^2 \cdot k'$	$\mathfrak{m}$	k	k'	$\mathfrak{m}^2 \cdot k$	$\mathfrak{m}^2 \cdot k'$
0	∞	∞	0	0	2.1	1.677	0.7822	7.395	3.450
0.1	19.70	48.63	0.1970	0.4863	2.2	1.646	0.7421	7.966	3.592
0.2	9.166	16.73	0.3666	0.6691	2.3	1.618	0.7059	8.557	3.734
0.3	6.166	9.258	0.5549	0.8332	2.4	1.592	0.6730	9.168	3.877
0.4	4.771	6.185	0.7633	0.9896	2.5	1.568	0.6430	9.799	4.019
0.5	3.968	4.567	0.9920	1.142	2.6	1.546	0.6154	10.45	4.160
0.6	3.445	3.589	1.240	1.292	2.7	1.526	0.5902	11.12	4.303
0.7	3.082	2.936	1.510	1.439	2.8	1.507	0.5669	11.81	4.444
0.8	2.812	2.477	1.800	1.585	2.9	1.489	0.5453	12.52	4.586
0.9	2.604	2.137	2.110	1.731	3.0	1.473	0.5253	13.25	4.728
1.0	2.439	1.876	2.489	1.876	3.1	1.457	0.5068	14.01	4.870
1.1	2.306	1.678	2.700	2.021	3.2	1.443	0.4895	14.78	5.012
1.2	2.194	1.503	3.160	2.164	3.3	1.430	0.4732	15.57	5.154
1.3	2.102	1.365	3.552	2.307	3.4	1.417	0.4581	16.38	5.296
1.4	2.021	1.250	3.961	2.450	3.5	1.405	0.4439	17.21	5.437
1.5	1.951	1.153	4.389	2.595	3.6	1.394	0.4305	18.06	5.580
1.6	1.891	1.069	4.841	2.739	3.7	1.383	0.4179	18.93	5.721
1.7	1.838	0.9965	5.312	2.880	3.8	1.373	0.4060	19.82	5.863
1.8	1.791	0.9332	5.804	3.024	3.9	1.363	0.3948	20.73	6.005
1.9	1.749	0.8767	6.314	3.165	4.0	1.354	0.3841	21.67	6.145
2.0	1.711	0.8268	6.845	3.307	∞	1	0	∞	∞

Table 2.1.: Values of the functions k and k' calculated for 40 different values of $\mathfrak{m}$

The last figure was then struck out, and five-figure logarithms were employed in multiplying the four functions M_0, M_0', M_e, and $1 - M_e'$ by $\frac{\pi}{4}$, and by L, as well as

in reducing the right-hand member of Equation 2.105 to the form $k + \mathbf{i}\,k'$. The terms of the series according to Equation 2.113 were calculated to five places of decimals. That these series are sufficiently convergent to be employed when $\mathfrak{m} = 1.5$, might be presumed from the numerical values of the terms, and is confirmed by finding that they give $k = 1.952$, and $k' = 1.153$. For $\mathfrak{m} = 1.5$ and a few of the succeeding values, the second and third of the series according to Equation 2.113 were summed directly as far as $\frac{1}{\mathfrak{m}^5}$ inclusively, and the remainders were calculated from Equation 2.114. Two columns are annexed, which give the values of $\mathfrak{m}^2\,k$ and $\mathfrak{m}^2\,k'$, and exhibit the law of the variation of the two parts of the force $\mathscr{F}$, when the radius of the cylinder varies, the nature of the fluid and time of oscillation remaining unchanged. Four significant figures are retained in all the results.
The numerical calculation by means of Equation 2.103, Equation 2.104, and Equation 2.105 become very laborious when many values of the functions are required. The difficulty of the calculation increases with the value of $\mathfrak{m}$ for two reasons, first, the calculation of the functions M_0, &c. becomes longer, and secondly, the moduli of the numerator and denominator of the fraction in the right-hand member of Equation 2.105 go on decreasing, so that greater and greater accuracy is required in the calculation of the functions M_0, &c., and of the products LM_0, &c., in order to ensure a given degree of accuracy in the result. The calculation by the descending series according to Equation 2.113 is on the contrary very easy.
It will be found that the first differences of $\mathfrak{m}^2\,k'$ and of $\mathfrak{m}^2 \cdot (k - 1)$ are nearly constant, except near the very beginning of the table. Hence in the earlier part of the table the value of k or k' for a value of $\mathfrak{m}$ not found in the table will be best got by finding $\mathfrak{m}^2\,k - \mathfrak{m}^2$ or m^2k' by interpolation, and thence passing to the value of k or k'. Very near the beginning of the table, interpolation would not succeed, but in such a case recourse may be had to Equation 2.103, Equation 2.104, and Equation 2.105, the calculation of which is comparatively easy when $\mathfrak{m}$ is small. It did not seem worthwhile to extend the table beyond $\mathfrak{m} = 4$, because when $\mathfrak{m}$ is greater than 4, the series according to Equation 2.113 are so rapidly convergent that k and k' may be calculated to a sufficient degree of accuracy with extreme facility.

38. Let us now examine the progress of the functions k and k'.
When $\mathfrak{m}$ is very small, we may neglect the powers of $\mathfrak{m}$ in the numerator and denominator of the fraction in the right-hand member of Equation 2.105, retaining only the logarithms and the constant terms. We thus get

$$k + \mathbf{i}\, k' = 1 - \frac{\mathfrak{m}^{-2}\,\mathbf{i}}{L - \frac{\pi}{4}\,\mathbf{i}}\,,$$

whence

$$\left.\begin{aligned} \mathfrak{m}^2 \cdot (k-1) &= \frac{\frac{\pi}{4}}{L^2 + \left(\frac{\pi}{4}\right)^2}\,, \\ \mathfrak{m}^2\, k' &= \frac{-L}{L^2 + \left(\frac{\pi}{4}\right)^2}\,, \\ &\vdots \end{aligned}\right\} \tag{2.115}$$

L being given by Equation 2.102 and Equation 2.104, or Equation 2.104 and Equation 2.106. When $\mathfrak{m}$ vanishes, L, which involves the logarithm of $\frac{1}{\mathfrak{m}}$, becomes infinite, but ultimately increases more slowly than if it varied as $\mathfrak{m}$ affected with any negative index however small. Hence it appears from Equation 2.115, that $k-1$ and k' are expressed by $\frac{1}{\mathfrak{m}^2}$ multiplied by two functions of $\mathfrak{m}$ which, though they ultimately vanish with $\mathfrak{m}$, decrease very slowly, so that a considerable change in $\mathfrak{m}$ makes but a small change in these functions. Now when the radius a of the cylinder varies, everything else remaining the same, $\mathfrak{m}$ varies as a, and in general the parts of the force $\mathscr{F}$ on which depend the alteration in the time of vibration, and the diminution in the arc of oscillation, vary as a^2k, a^2k', respectively. Hence in the case of a cylinder of small radius, such as the wire used to support a sphere in a pendulum experiment, a considerable change in the radius of the cylinder produces a comparatively small change in the part of the alteration in the time and arc of vibration which is due to the resistance experienced by the wire. The simple Equation 2.115 are accurate enough for the fine wires usually employed in such experiments if the theory itself be applicable; but reasons will presently be given for regarding the application of the theory to such fine wires as extremely questionable.
From $\mathfrak{m} = 0.3$ or 0.4 to the end of the table, the first differences of each of the functions $\mathfrak{m}^2 \cdot (k-1)$ and $\mathfrak{m}^2\, k'$ remain nearly constant. Hence for a considerable range of values of $\mathfrak{m}$, each of the functions may be expressed pretty accurately by $A + B\,\mathfrak{m}$. When $\mathfrak{m}$ is at all large, the first two terms in the 2nd and 3rd of the Equation 2.113 will give k and k' with considerable accuracy, because, independently of the decrease of the successive quantities $\frac{1}{\mathfrak{m}}$, $\frac{1}{\mathfrak{m}^2}$, $\frac{1}{\mathfrak{m}^3}$, ..., the coefficients of $\frac{1}{\mathfrak{m}}$ and $\frac{1}{\mathfrak{m}^2}$ are considerably larger than those of several of the succeeding powers. If we

neglect in these formulæ the terms after u_2, we get

$$k = 1 + \sqrt{2} \cdot \frac{1}{\mathfrak{m}}\,,$$
$$k' = \sqrt{2} \cdot \frac{1}{\mathfrak{m}} + \frac{1}{2} \cdot \frac{1}{\mathfrak{m}^2}\,.$$

It may be remarked that these approximate expressions, regarded as functions of the radius a, have precisely the same form as the exact expressions obtained for a sphere, the coefficients only being different.

2.4. Determination of the motion of a fluid about a sphere which moves uniformly with a small velocity. Justification of the application of the solutions obtained in section 2.2 and section 2.3 to cases in which the extent of oscillation is not small in comparison with the radius of the sphere or cylinder. Discussion of a difficulty which presents itself with reference to the uniform motion of a cylinder in a fluid.

39. Let a sphere move in a fluid with a uniform velocity V, its centre moving in a right line; and let the rest of the notation be the same as in section 2.2. Conceive a velocity equal and opposite to that of the sphere impressed both on the sphere and on the fluid, which will not affect the relative motion of the sphere and fluid, and will reduce the determination of the motion of the fluid to a problem of steady motion. Then we have for the equations of condition

$$R = 0\,, \quad \Theta = 0\,, \quad \text{when} \;\; r = a\,; \tag{2.116}$$

$$R = -V \cdot \cos(\theta)\,, \quad \Theta = V \cdot \sin(\theta)\,, \quad \text{when} \;\; r = \infty\,. \tag{2.117}$$

The equations of condition, as well as the equations of motion, may be satisfied by supposing ψ to have the form $\sin^2(\theta) \cdot f(r)$. We get from Equation 11.20', by the same process as that by which Equation 2.33 and Equation 2.34 were obtained,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2}{r^2}\right)^2 \cdot f(r) = 0\,, \tag{2.118}$$

the only difference being that in the present case Equation 11.20' cannot be replaced by the two Equation 2.22 and Equation 2.23, which become identical, inasmuch as the velocity of the fluid is independent of the time.
The integral of Equation 2.118 is

$$f(r) = Ar^{-1} + Br + Cr^2 + Dr^4\,, \tag{2.119}$$

which gives

$$R = \frac{1}{r^2 \cdot \sin(\theta)} \cdot \frac{\mathrm{d}\psi}{\mathrm{d}\theta} = 2 \cdot \cos(\theta) \cdot \left(\frac{A}{r^3} + \frac{B}{r} + C + Dr^2\right)\,,$$
$$\Theta = -\frac{1}{r \cdot \sin(\theta)} \cdot \frac{\mathrm{d}\psi}{\mathrm{d}r} = \sin(\theta) \cdot \left(\frac{A}{r^3} - \frac{B}{r} - 2C - 4Dr^2\right)\,.$$

The first of the equations of condition according to Equation 2.117 requires that

$$D = 0\,, \quad C = -\frac{1}{2} \cdot V\,. \tag{2.120}$$

It is particularly to be remarked that inasmuch as the two arbitrary constants C and D are determined by the first of the conditions according to Equation 2.117, none remain whereby to satisfy the second. Nevertheless it happens that the second of these conditions leads to precisely the same Equation 2.120 as the first. The equations of condition according to Equation 2.116 give

$$A = -\frac{1}{4} \cdot Va^3\,, \quad B = \frac{3}{4} \cdot Va\,;$$

whence[15]

$$\psi = \frac{1}{4} \cdot Va^2 \cdot \left(-\frac{2r^2}{a^2} + \frac{3r}{a} - \frac{a}{r}\right) \cdot \sin^2(\theta)\,, \tag{2.121}$$

$$R = -V \cdot \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right) \cdot \cos(\theta)\,, \tag{2.122}$$

$$\Theta = V \cdot \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) \cdot \sin(\theta)\,. \tag{2.123}$$

If now we wish to obtain the solution of the problem in its original shape, in which the sphere is in motion and the fluid at rest, except so far as it is disturbed by the sphere, we have merely to add $V \cdot \cos(\theta)$, $-V \cdot \sin(\theta)$ and $\frac{1}{2} \cdot Vr^2 \cdot \sin^2(\theta)$ to the

[15] I have already had occasion, in treating of another subject, to publish the solution expressed by this equation, which I had obtained as a limiting case of the problem of a ball pendulum. See *Philosophical Magazine* for May 1848, p. 343, Part IV.

expressions for R, Θ, and ψ. We get from Equation 2.121

$$\psi = \frac{1}{4} \cdot Va^2 \cdot \left(\frac{3r}{a} - \frac{a}{r} \right) \cdot \sin^2(\theta) \,. \tag{2.124}$$

40. Let us now return to the problem of section 2.2; let us suppose the time of oscillation to increase indefinitely, and examine what Equation 2.40 becomes in the limit.

When τ becomes infinite, n, and therefore m, vanishes; the expression within brackets in Equation 2.40 takes the form $\infty - \infty$, and its limiting value is easily found by the ordinary methods. We must retain the m^2 in the coefficient of t, because t is susceptible of unlimited increase. We get in the limit

$$\psi = \frac{1}{4} a^2 c \cdot e^{\mu' m^2 t} \cdot \left(\frac{3r}{a} - \frac{a}{r} \right) \cdot \sin^2(\theta) \,. \tag{2.125}$$

If now we put V for $\frac{\mathrm{d}\xi}{\mathrm{d}t}$, the velocity of the sphere, we get from Equation 2.39, $c \cdot e^{\mu' m^2 t} = V$. After substituting in Equation 2.125, the equation will remain unchanged when we pass from the symbolical to the real values of ψ and V, and thus Equation 2.125 will be reduced to Equation 2.124.

41. It appears then that by supposing the rate of alteration of the velocity of the sphere to decrease indefinitely, we obtain from the solution of the problem of section 2.2 the same result as was obtained in Art. 39, page 179, by treating the motion as steady. As yet, however, the method of Art. 40, page 181 is subject to a limitation from which that of Art. 39, page 179 is free. In obtaining Equation 2.40, it was supposed that the maximum excursion of the sphere was small in comparison with its radius. Retaining this restriction while we suppose τ to become very large, we are obliged to suppose c to become very small, so that the velocity of the sphere is not merely so small that we may neglect terms depending upon its square, a restriction to which Art. 39, page 179 is also subject, but so extremely small that the space passed over by the sphere in even a long time is small in comparison with its radius.

We have seen, however, that on supposing τ very large in Equation 2.40 we obtain a result identical with Equation 2.124, not merely a result with which Equation 2.124 becomes identical when the restriction above mentioned is introduced. This leads to the supposition that the solution expressed by Equation 2.40 is in fact more general than would appear from the way in which it was obtained. That such is really the case may be shewn by a slight modification of the analysis. Instead of referring the fluid to axes fixed in space, refer it to axes originating at the centre of the sphere, and moveable with it. In the general equations of motion, the terms which contain differential coefficients taken with respect to the co-ordinates will remain unchanged, inasmuch as they represent the very same limiting ratios as before: it is only those in which differentiation with respect to t occurs that will be

altered. If $\frac{\mathrm{d}'}{\mathrm{d}t}$ be the symbol of differentiation with respect to t when the fluid is referred to the moveable axes, we shall have

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}'}{\mathrm{d}t} - \frac{\mathrm{d}\xi}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}x};$$

but the terms arising from $\frac{\mathrm{d}\xi}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}x}$ are of the order of the square of the velocity, and are therefore to be neglected. Hence the general equations have the same form whether the fluid be referred to the fixed or moveable axes. But on the latter supposition the equations of condition according to Equation 2.30 become rigorously exact. Hence Equation 2.40 gives correctly the solution of the problem, independently of the restriction that the maximum excursion of the sphere be small compared with its radius, provided we suppose the polar co-ordinates r and θ be measured from the centre of the sphere in its actual, not its mean position. Similar remarks apply to the problem of the cylinder. Moreover, in the case of a sphere oscillating within a concentric spherical envelope, it is not necessary, in order to employ the solution obtained in section 2.2, that the maximum excursion of the sphere be small compared with its radius; it is sufficient that it be small compared with the radius of the envelope.

These are points of great importance, because the excursions of an oscillating sphere in a pendulum experiment are not by any means extremely small compared with the radius of the sphere; and in the case of a narrow cylinder, such as the suspending wire, so far from the maximum excursion being small compared with the radius of the cylinder, it is, on the contrary, the radius which is small compared with the maximum excursion.

42. Let us now return to the case of the uniform motion of a sphere. In order to obtain directly the expression for the *resistance* of the fluid, it would be requisite first to find p, then to get P_r and T_θ from Equation 2.46, or at least to get the values of these functions for $r = a$, and lastly to substitute these in Equation 2.47 and perform the integration. We should obtain p by integrating the expression for $\mathrm{d}p$ got from Equation 2.16 and Equation 2.17. It would be requisite first to express u and q in terms of ψ, then to transform the expression for $\mathrm{d}p$ so as to involve polar co-ordinates, and then substitute for ψ its value given by Equation 2.121; or else to express the right-hand member of Equation 2.121 by the co-ordinates x, ϖ, and substitute in the expression for $\mathrm{d}p$.[16] We have seen, however, that the results

[16]The Equation 2.16 and Equation 2.17 give, after a troublesome transformation to polar co-ordinates,

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mu}{r^2 \cdot \sin(\theta)}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\sin(\theta)}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\sin(\theta)}\frac{\mathrm{d}}{\mathrm{d}\theta} - \frac{\rho}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}\right)\psi, \qquad (\gamma)$$

$$\frac{\mathrm{d}p}{\mathrm{d}\theta} = -\frac{\mu}{\sin(\theta)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\sin(\theta)}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\sin(\theta)}\frac{\mathrm{d}}{\mathrm{d}\theta} - \frac{\rho}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}\right)\psi. \qquad (\delta)$$

The expression for $\mathrm{d}p$ got from these equations is an exact differential by virtue of the equation

applicable to uniform motion may be deduced as limiting cases of those which relate to oscillatory motion, and consequently, we may make use of the expression for $\mathscr{F}$ already worked out. Writing V for $c \cdot e^{\mathbf{i}\,nt}$ in the first equation of Art. 20, page 157, where c is the maximum excursion of similarly situated points of the fluid and expressing m in terms of n according to Equation 2.32, and then making n vanish, we get

$$\boxed{-\mathscr{F} = 6\pi\mu'\rho a V} \tag{2.126}$$

and $-\mathscr{F}$ is the resistance required.

This equation may be employed to determine the terminal velocity of a sphere ascending or descending in a fluid, provided the motion be so slow that the square of the velocity may be neglected. It has been shewn experimentally by COULOMB[17], that in the case of very slow motions, the resistance of a fluid depends partly on the square and partly on the first power of the velocity. Equation 2.126 determines, in the particular case of a sphere, that part of the whole resistance which depends on the first power of the velocity, even though the part which depends on the square of the velocity be not wholly insensible.

It is particularly to be remarked, that according to Equation 2.126, the resistance varies not as the surface but as the radius of the sphere, and consequently the quotient of the resistance divided by the mass increases in a higher ratio, as the radius diminishes, than if the resistance varied as the surface. Accordingly, fine powders remain nearly suspended in a fluid of widely different specific gravity.

43. When the motion is so slow that the part of the resistance which depends on the square of the velocity may be neglected, we have, supposing V to be the terminal velocity,

$$-\mathscr{F} = \frac{4}{3}\pi g \cdot (\sigma - \rho) \cdot a^3\,,$$

where g is the force of gravity, and σ, which is supposed greater than ρ, the density

which determines ψ; and in the problems considered in section 2.2 and in the present Section ψ has the form $\Psi \cdot \sin^2(\theta)$, where Ψ is independent of θ. Hence we get from Equation δ, by integrating partially with respect to θ,

$$p = \mu \cdot \cos(\theta) \cdot \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2}{r^2} - \frac{\rho}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}\right)\Psi\,. \tag{ϵ}$$

It is unnecessary to add an arbitrary function of r, because if $\lambda(r)$ be such a function which we suppose added to the right-hand member of Equation ϵ, we must determine λ by substituting in Equation γ. The resulting expression for $\lambda'(r)$ cannot contain θ, inasmuch as the expression for $\mathrm{d}p$ is an exact differential, but it is composed of terms which all involve $\cos(\theta)$ as a factor, and therefore we know, without working out, that these terms must destroy one another. Hence $\lambda(r)$ must be constant, or at most be a function of t, which we may suppose included in Π. $\lambda(r)$ will in fact be equal to zero if Π be the equilibrium pressure at the depth at which $\int g\,\mathrm{d}z'$ vanishes.

[17] *Mémoires de l'Institut*, Tom. III. p. 246.

of the sphere. Substituting in Equation 2.126 we get

$$\boxed{V = \frac{2g}{9\mu'} \cdot \left(\frac{\sigma}{\rho} - 1\right) \cdot a^2}. \tag{2.127}$$

Let us apply this equation to determine the terminal velocity of a globule of water forming part of a cloud. Putting $g = 386\ \frac{\text{in}}{\text{sec}}$, $\mu' = (0.116)^2 \frac{\text{in}^2}{\text{sec}}$, an inch being the unit of length, and supposing $\left(\frac{\sigma}{\rho} - 1\right) = 1000$, in order to allow a little for the rarity of the air at the height of the cloud, we get $V = 6372 \times 1000 \cdot a^2$. Thus, for a globule the one thousandth of an inch in diameter, we have $V = 1.593$ inch per second. For a globule the one ten thousandth of an inch in diameter, the terminal velocity would be a hundred times smaller, so as not to amount to the one sixtieth part of an inch per second.
We may form a very good judgment of the magnitude of that part of the resistance which varies as the square of the velocity, and which is the only kind of resistance that could exist if the pressure were equal in all directions, by calculating the numerical value of the resistance according to the common theory, imperfect though it be. It follows from this theory that if h be the height due to the velocity V, the resistance is to the weight as $3\rho h$ to $8\sigma a$. For $V = 1.593$ inch per second, the resistance is not quite the one four hundredth part of the weight; and for a sphere only the one ten thousandth of an inch in diameter, moving with the velocity calculated from Equation 2.127, the ratio of the resistance to the weight would be ten times as small. The terminal velocities of the globules calculated from the common theory would be 32.07 and 10.14 inches per second, instead of only 1.593 and 0.01593 inch. It appears then that the apparent suspension of the clouds is mainly due to the internal friction of air.

44. The resistance to the globule has here been determined as if the globule were a solid sphere. In strictness, account ought to be taken of the relative motion of the fluid particles forming the globule itself. Although it may readily be imagined that no material change would thus be made in the numerical result, it may be worth while to point out the mode of solution of the problem. Suppose the globule preserved in a strictly spherical shape by capillary attraction, which will very nearly indeed be the case. Conceive a velocity equal and opposite to that of the globule impressed both on the globule and on the surrounding fluid, which will reduce the problem to one of steady motion. Let ψ_1, &c. refer to the fluid forming the globule, and assume $\psi_1 = f_1(r) \cdot \sin^2(\theta)$. Then we get on changing the constants in Equation 2.119

$$f_1(r) = \frac{A_1}{r} + B_1 r + C_1 r^2 + D_1 r^4 .$$

The arbitrary constants A_1, B_1 vanish by the condition that the velocity shall not become infinite at the centre. There remain the two arbitrary constants C_1, D_1 to

be determined, in addition to those which appeared in the former problem. But we have now four instead of two equations of condition which have to be satisfied at the surface of the sphere, which are that

$$\left.\begin{array}{ll} R = 0\,, & \Theta = \Theta_1\,, \\ R_1 = 0\,, & T_\theta = T_{1\theta}\,, \end{array}\right\} \tag{2.128}$$

where $r = a$.
We shall thus have the same number of arbitrary constants as conditions to be satisfied. Now $T_{1\theta}$ will involve μ_1 as a coefficient, just as T_θ involves $\mu'\rho$ or μ; and μ_1, which refers to water, is much larger than μ, which refers to air, although μ' is larger than μ_1'. Hence the results will be nearly the same as if we had taken $\mu_1 = \infty$, or regarded the sphere as solid.
If, however, instead of a globule of liquid descending in a gas we have a very small bubble ascending in a liquid, we must not treat the bubble as a solid sphere. We may in this case also neglect the motion of the fluid forming the sphere, but we have now arrived at the other extreme case of the general problem, and the two equations of condition which have to be satisfied at the surface of the sphere are that $R = 0$ and $T_\theta = 0$ when $r = a$, instead of $R = 0$ and $\Theta = 0$, when $r = a$.
The equation of condition $T_\theta = 0$ which applies to a bubble, as well as the fourth of Equation 2.128, will not be the true equations, if forces arising from internal friction exist in the superficial film of a fluid which are of a different order of magnitude from those which exist throughout the mass. At the end of the memoir already referred to, Coulomb states that in very slow motions the resistance of bodies not completely immersed in a liquid is much greater than that of bodies wholly immersed, and promises to communicate a second memoir in continuation of the first. This memoir, so far as I can find out, has never appeared. Should the existence of such forces in the superficial film of a liquid be made out, the results deduced from the theory of internal friction will be modified in a manner analogous to that in which the results deduced from the common principles of hydrostatics are modified by capillary attraction. It may be remarked that we have nothing to do with forces of this kind in considering the motion of pendulums in air, or even in considering the oscillations of a sphere in water, except as regards the very minute fraction of the whole effect which relates to the resistance experienced by the suspending wire in the immediate neighbourhood of the free surface.
It may readily be seen that the effect of a set of forces in the superficial film of a liquid offering a peculiar resistance to the relative motion of the particles would be, to make the resistance of a gas to a descending globule agree still more clearly with the result obtained by regarding the globule as solid, while the resistance experienced by an ascending bubble would be materially increased, and made to approach to that which would belong to a solid sphere of the same size without mass, or more strictly, with a mass only equal to that of the gas forming the bubble.

Possibly the determination of the velocity of ascent of very small bubbles may turn out to be a good mode of measuring the amount of friction in the superficial film of a liquid, if it be true that forces of this kind have any existence. But any investigation relating to such a subject would at present be premature.

45. Let us now attempt to determine the uniform motion of a fluid about an infinite cylinder. Employing the notation of section 2.3, and reducing the problem to one of steady motion as in Art. 39, page 179, we obtain the same equations of condition according to Equation 2.116 and Equation 2.117, as in the case of the sphere. Assuming $\chi = \sin(\theta) \cdot F(r)$, and substituting in the equation obtained from Equation 2.69 by transforming to polar co-ordinates and leaving out the terms which involve $\frac{\mathrm{d}}{\mathrm{d}t}$, we get

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2}\right)^2 \cdot F(r) = 0\,. \tag{2.129}$$

The integral of this equation may readily be obtained by multiplying the last term of the operating factor by $(1+\delta)^2$, integrating the transformed equation, and then making δ vanish. It is

$$F(r) = \frac{A}{r} + Br + Cr\log(r) + Dr^3\,, \tag{2.130}$$

which gives

$$R = \frac{1}{r} \cdot \frac{\mathrm{d}\chi}{\mathrm{d}\theta} = \left(\frac{A}{r^2} + B + C\log(r) + Dr^2\right) \cdot \cos(\theta)\,,$$
$$\Theta = -\frac{\mathrm{d}\chi}{\mathrm{d}r} = \left(\frac{A}{r^2} - B - C - C\log(r) - 3Dr^2\right) \cdot \sin(\theta)\,.$$

The first of the equations of condition in Equation 2.117 requires that

$$C = 0\,, \quad D = 0\,, \quad B = -V\,,$$

which also satisfies the second. We have thus only one arbitrary constant left whereby to satisfy the two equations of condition according to Equation 2.116, and the same value of A will not satisfy these two equations.

46. It appears then that the supposition of steady motion is inadmissible. It will be remembered that, in the case of the sphere, the solution of the problem was only possible because it so happened that the values of two arbitrary constants determined by satisfying the first of the equations of condition in Equation 2.117 satisfied also the second, which indicates that the solution was to a certain extent tentative. We have evidently a right to conceive a sphere or infinite cylinder to exist at rest in an infinite mass of fluid also at rest, to suppose the sphere or cylinder

to be then moved with a uniform velocity V, and to propose for determination the motion of the fluid at the end of the time t. But we have no right to assume that the motion approaches a permanent state as t increases indefinitely. We may follow either of two courses. We may proceed to solve the general problem in which the sphere or cylinder is supposed to move from rest, and then examine what results we obtain by supposing t to increase indefinitely, or else we may assume *for trial* that the motion is steady, and proceed to inquire whether we can satisfy all the conditions of the problem on this supposition. The former course would have the disadvantage of requiring a complicated analysis for the sake of obtaining a comparatively simple result, and it is even possible that the solution of the problem might baffle us altogether; but if we adopt the latter course, we must not forget that the equations with which we work are only provisional.
It might be objected that the impossibility of satisfying the conditions of the problem on the hypothesis of steady motion arose from our assumption that $\sin(\theta)$ was a factor of χ, the other factor being independent of θ. This however is not the case. For, for given values of r and t, χ is a finite function of θ from $\theta = 0$ to $\theta = \pi$. We have a right to suppose χ to vanish at any point of the axis of x positive that we please; and if we suppose χ to vanish at one such point, it may be shewn as in the note to Art. 15, page 151, that χ will vanish at all points of the axis of x positive or x negative. Hence χ may be expanded in a convergent series of sines of θ and its multiples; and since χ and its derivatives with respect to θ alter continuously with θ, the expansions of the derivatives will be got by direct differentiation[18]. This being true for all other pairs of values of r and t, χ can in general be expanded in a convergent series of sines of θ and its multiples; but the coefficients, instead of being constant, will be functions of r and t, or in the particular case of steady motion, functions of r alone. Now a very slight examination of the general equations will suffice to shew that the coefficients of the sines of the different multiples of θ remain perfectly independent throughout the whole process, and consequently had we employed the general expansion, we should have been led to the very same conclusions which have been deduced from the assumed form of χ.

47. If we take the impossibility of the existence of a limiting state of motion, which has just been established, in connexion with the results obtained in section 2.3, we shall be able to understand the general nature of the motion of the fluid around an infinite cylinder which is at first at rest, and is then moved on indefinitely with a uniform velocity.
The fluid being treated as incompressible, the first motion which takes place is impulsive. Since the terms depending on the internal friction will not appear in the calculation of this motion, we may employ the ordinary equations of hydrodynamics.

[18] See a paper *On the Critical Values of the Sums of Periodic Series.* Camb. Phil. Trans. Vol. VIII. p. 533.

The result, which is easily obtained, is[19]

$$R\,\mathrm{d}r + \Theta r\,\mathrm{d}\theta = \mathrm{d}\phi\,, \qquad \text{where } \phi = -\frac{Va^2}{r}\cdot\cos(\theta)\,. \tag{2.131}$$

As the cylinder moves on, it carries more and more of the fluid with it, in consequence of friction. For the sake of precision, let the quantity carried by the element $\mathrm{d}l$ of the cylinder be defined to be that which, moving with the velocity V, would have the same momentum in the direction of the motion that is actually possessed by the elementary portion of fluid which is contained between two parallel infinite planes drawn perpendicular to the axis of the cylinder, at an interval $\mathrm{d}l$, the particles composing which are moving with velocities that vary from V to zero in passing from the surface outwards. The pressure of the cylinder on the fluid continually tends to increase the quantity of fluid which it carries with it, while the friction of the fluid at a distance from the cylinder continually tends to diminish it. In the case of a sphere, these two causes eventually counteract each other, and the motion becomes uniform. But in the case of a cylinder, the increase in the quantity of fluid carried continually gains on the decrease due to the friction of the surrounding fluid, and the quantity carried increases indefinitely as the cylinder moves on. The rate at which the quantity carried is increased, decreases continually, because the motion of the fluid in the neighbourhood of the cylinder becomes more and more nearly a simple motion of translation equal to that of the cylinder itself, and therefore the rate at which the quantity of fluid carried is increased would become smaller and smaller, even were no resistance offered by the surrounding fluid. The correctness of this explanation is confirmed by the following considerations. Suppose that $F(r)$ had been given by the equation

$$F(r) = \frac{A}{r} + Br + Cr^{1-\delta} + Dr^3$$

instead of Equation 2.130, δ being a small positive quantity. On this supposition it would have been possible to satisfy all the equations of condition, and therefore steady motion would have been possible. By determining the arbitrary constants,

[19] According to these equations, the fluid flows past the surface of the cylinder with a finite velocity. At the end of the small time t after the impact, the friction has reduced the velocity of the fluid in contact with the cylinder to that of the cylinder itself, and the tangential velocity alters very rapidly in passing from the surface outwards. At a small distance s from the surface of the cylinder, the relative velocity of the fluid and the cylinder, in a direction tangential to the surface, is a function of the independant variables t', s, which vanishes with s for any given value of t', however small, but which for any given value of s, however small, approaches indefinitely to the quantity determined by Equation 2.131 as t vanishes. The communication of lateral motion is similar to the communication of temperature when the surface of a body has its temperature instantaneously raised or lowered by a finite quantity.

and substituting in χ, we should have obtained

$$\psi = a \cdot V \cdot \left\{ -\frac{\delta}{(2-\delta)} \frac{a}{r} - \frac{r}{a} + \frac{2}{(2-\delta)} \left(\frac{r}{a}\right)^{1-\delta} \right\} \cdot \sin(\theta) \,,$$

$$R = V \cdot \left\{ -\frac{\delta}{(2-\delta)} \left(\frac{a}{r}\right)^{2} - 1 + \frac{2}{(2-\delta)} \left(\frac{r}{a}\right)^{-\delta} \right\} \cdot \cos(\theta) \,,$$

$$\Theta = V \cdot \left\{ -\frac{\delta}{(2-\delta)} \left(\frac{a}{r}\right)^{2} + 1 - \frac{2 \cdot (1-\delta)}{(2-\delta)} \left(\frac{r}{a}\right)^{-\delta} \right\} \cdot \sin(\theta) \,.$$

Since δ is supposed to be extremely small, it follows from these expressions that when r is not greater than a moderate multiple of a, the velocities R, Θ are extremely small; but, however small be δ, we have only to go far enough from the cylinder in order to find velocities as nearly equal to $-V \cdot \cos(\theta)$, $+V \cdot \sin(\theta)$ as we please. But the distance from the cylinder to which we must proceed in order to find velocities R, Θ which do not differ from their limiting values $-V \cdot \cos(\theta)$, $+V \cdot \sin(\theta)$ by more than certain given quantities, increases indefinitely as δ decreases. Hence, restoring to the fluid and the cylinder the velocity V, we see that in the neighbourhood of the cylinder the motion of the fluid does not sensibly differ from a motion of translation, the same as that of the cylinder itself, while the distance to which the cylinder exerts a sensible influence in disturbing the motion of the fluid increases indefinitely as δ decreases.

48. When we have formed the equations of motion of a fluid on any particular dynamical hypothesis, it becomes a perfectly definite mathematical problem to determine the motion of the fluid when a given solid, initially at rest as well as the fluid, is moved in a given manner, or to discuss the character of the analytical solution in any extreme case proposed. It is quite another thing to enquire how far the principles which furnished the mathematical data of the problem may be modified in extreme cases, or what will be the nature of the actual motion in such cases. Let us regard in this point of view the case considered in the preceding article as a mathematical problem. When the quantity of fluid carried with the cylinder becomes considerable compared with the quantity displaced, it would seem that the motion must become unstable, in the sense in which the motion of a sphere rolling down the highest generating line of an inclined cylinder may be said to be unstable. But besides the instability, it may not be safe in such an extreme case to neglect the terms depending on the square of the velocity, not that they become unusually large in themselves, but only unusually large compared with the terms retained, because when the relative motions of neighbouring portions of the fluid become very small, the tangential pressures which arise from friction become very small likewise.

Now the general character of the motion must be nearly the same whether the velocity of the cylinder be constant, or vary slowly with the time, so that it does not vary materially when the cylinder passes through a space equal to a small multiple

of its radius. To return to the problem considered in section 2.3, it would seem that when the radius of the cylinder is very small, the motion which would be expressed by the formulæ of that Section would be unstable. This might very well be the case with the fine wires used in supporting the spheres employed in pendulum experiments. If so, the quantity of fluid carried by the wire would be diminished, portions being continually left behind and forming eddies. The resistance to the wire would on the whole be increased, and would moreover approximate to a resistance which would be a function of the velocity. Hence, so far as depends on the wire, the arc of oscillation would be more affected by the resistance of the air than would follow from the formulæ of section 2.3. Whether the effect on the time of oscillation would be greater or less than that expressed by the formulæ is difficult to say, because the increase of resistance would tend to increase the effect on the time of vibration, while on the other hand the approximation of the law of resistance to that of a function of the velocity would tend to diminish it.

2.5. On the effect of internal friction in causing the motion of a fluid to subside. Application to the case of oscillatory waves

49. We have already had instances of the effect of friction in causing a gradual subsidence in the motion of a solid oscillating in a fluid ; but a result may easily be obtained from the equations of motion in their most general shape, which shews very clearly the effect of friction in continually consuming a portion of the work of the forces acting on the fluid.

Let P_1, P_2, P_3 be the three normal, and T_1, T_2, T_3 the three tangential pressures in the direction of three rectangular planes parallel to the co-ordinate planes, and let D be the symbol of differentiation with respect to t when the particle and not the point of space remains the same. Then the general equations applicable to a heterogeneous fluid, (the Equation 2.10 of my former paper,) are

$$\rho \cdot \left(\frac{\mathrm{D}u}{\mathrm{D}t} - X \right) + \frac{\mathrm{d}P_1}{\mathrm{d}x} + \frac{\mathrm{d}T_3}{\mathrm{d}y} + \frac{\mathrm{d}T_2}{\mathrm{d}z} = 0 \,, \tag{2.132}$$

with the two other equations which may be written down from symmetry. The pressures P_1, &c. are given by the equations

$$\left.\begin{aligned} P_1 &= p - 2\mu \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}x} - \delta \right) \,, \\ T_1 &= -\mu \cdot \left(\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y} \right) \,, \end{aligned}\right\} \tag{2.133}$$

and four other similar equations. In these equations

$$3\delta = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z}\,. \tag{2.134}$$

At the end of the time t let V be the *vis viva* of a limited portion of the fluid, occupying the space which lies inside the closed surface S, and let $V + \mathrm{D}V$ be the *vis viva* of the same mass at the end of the time $t + \mathrm{D}t$. Then

$$\begin{aligned} V &= \iiint \rho \left(u^2 + v^2 + w^2\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,, \\ \mathrm{D}V &= 2\mathrm{D}t \cdot \iiint \rho \left(u\frac{\mathrm{D}u}{\mathrm{D}t} + v\frac{\mathrm{D}v}{\mathrm{D}t} + w\frac{\mathrm{D}w}{\mathrm{D}t}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,, \end{aligned} \tag{2.135}$$

the triple integrals extending throughout the space bounded by S. Substituting now for $\frac{\mathrm{D}u}{\mathrm{D}t}$, &c. their values given by the equations of the system according to Equation 2.132, we get

$$\begin{aligned} DV &= 2Dt \cdot \iiint \rho \cdot (uX + vY + wZ)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &- 2Dt \cdot \iiint \left\{ u \cdot \left(\frac{\mathrm{d}P_1}{\mathrm{d}x} + \frac{\mathrm{d}T_3}{\mathrm{d}y} + \frac{\mathrm{d}T_2}{\mathrm{d}z}\right) + v \cdot \left(\frac{\mathrm{d}P_2}{\mathrm{d}y} + \frac{\mathrm{d}T_1}{\mathrm{d}z} + \frac{\mathrm{d}T_3}{\mathrm{d}x}\right) \right. \\ &\left. + w \cdot \left(\frac{\mathrm{d}P_3}{\mathrm{d}z} + \frac{\mathrm{d}T_2}{\mathrm{d}x} + \frac{\mathrm{d}T_1}{\mathrm{d}y}\right) \right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \end{aligned} \tag{2.136}$$

The first part of this expression is evidently twice the work, during the time $\mathrm{D}t$, of the external forces which act all over the mass. The second part becomes after integration by parts

$$\begin{aligned} &- 2Dt \cdot \iint (uP_1 + vT_3 + wT_2)\,\mathrm{d}y\,\mathrm{d}z - 2Dt \cdot \iint (vP_2 + wT_1 + uT_3)\,\mathrm{d}z\,\mathrm{d}x \\ &- 2Dt \cdot \iint (wP_3 + uT_2 + vT_1)\,\mathrm{d}x\,\mathrm{d}y \\ &+ 2Dt \cdot \iiint \left\{ \frac{\mathrm{d}u}{\mathrm{d}x}P_1 + \frac{\mathrm{d}v}{\mathrm{d}y}P_2 + \frac{\mathrm{d}w}{\mathrm{d}z}P_3 + \left(\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y}\right) \cdot T_1 + \left(\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z}\right) \cdot T_2 \right. \\ &\left. + \left(\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}x}\right) \cdot T_3 \right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \end{aligned}$$

The double integrals in this expression are to be extended over the whole surface S. If $\mathrm{d}S$ be an element of this surface, l', m', n' the direction-cosines of the normal drawn outwards at $\mathrm{d}S$, we may write $l'\,\mathrm{d}S$, $m'\,\mathrm{d}S$, $n'\,\mathrm{d}S$ for $\mathrm{d}y\,\mathrm{d}z$, $\mathrm{d}z\,\mathrm{d}x$, $\mathrm{d}x\,\mathrm{d}y$.

The second part of DV thus becomes

$$-2Dt\cdot\iint\{u\cdot(l'P_1+m'T_3+n'T_2)+v\cdot(m'P_2+n'T_1+l'T_3)$$
$$+w\cdot(n'P_3+l'T_2+m'T_1)\}\,\mathrm{d}S\,.$$

The coefficients of u, v, w in this expression are the resolved parts, in the direction of x, y, z, of the pressure on a plane in the direction of the elementary surface dS, whence it appears that the expression itself denotes twice the work of the pressures applied to the surface of the portion of fluid that we are considering.
On substituting for P_1, &c. their values given by Equation 2.133 and Equation 2.134, we get for the last part of DV

$$+2Dt\cdot\iiint p\cdot\left(\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\mathrm{d}v}{\mathrm{d}y}+\frac{\mathrm{d}w}{\mathrm{d}z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$-2Dt\cdot\iiint\mu\cdot\left\{2\cdot\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+2\cdot\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2+2\cdot\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2\right.$$
$$-\frac{2}{3}\cdot\left(\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\mathrm{d}v}{\mathrm{d}y}+\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2$$
$$\left.+\left(\frac{\mathrm{d}v}{\mathrm{d}z}+\frac{\mathrm{d}w}{\mathrm{d}y}\right)^2+\left(\frac{\mathrm{d}w}{\mathrm{d}x}+\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2+\left(\frac{\mathrm{d}u}{\mathrm{d}y}+\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

In this expression p denotes, in the case of an elastic fluid, the pressure statically corresponding to the density which actually exists about the point whose co-ordinates are x, y, z, and the part of the expression which contains p denotes twice the work converted into *vis viva* in consequence of internal expansions, and arising from the forces on which the elasticity depends. The last part of the expression is essentially negative, or at least cannot be positive, and can only vanish in one very particular case. It denotes the *vis viva* consumed, or twice the work lost in the system during the time dt, in consequence of internal friction. According to the very important theory of Mr Joule, which is founded on a set of most striking and satisfactory experiments, the work thus apparently lost is in fact converted into heat, at such a rate, that the work expressed by the descent of 772 lbs through one foot, supplies the quantity of heat required to raise 1 lb. of water through 1° of Fahrenheit's thermometer.

50. The triple integral containing μ can only vanish when the differential coefficients of u, v, w satisfy the four following equations,

$$\left.\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y} = \frac{\mathrm{d}w}{\mathrm{d}z}\,, \\ \frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y} = 0\,, \\ \frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z} = 0\,, \\ \frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}x} = 0\,. \end{aligned}\right\} \tag{2.137}$$

These equations give immediately the following expressions for the differentials of u, v, w, in which the co-ordinates alone are supposed to vary, the time being constant:

$$\left.\begin{aligned} \mathrm{d}u = \delta\,\mathrm{d}x - \omega'''\,\mathrm{d}y + \omega''\,\mathrm{d}z\,, \\ \mathrm{d}v = \delta\,\mathrm{d}y - \omega'\,\mathrm{d}z + \omega'''\,\mathrm{d}x\,, \\ \mathrm{d}w = \delta\,\mathrm{d}z - \omega''\,\mathrm{d}x + \omega'\,\mathrm{d}y\,. \end{aligned}\right\} \tag{2.138}$$

In these equations δ, ω', ω'', ω''' are certain functions of which the forms are defined by the Equation 2.138, but need not at present be considered. It follows from Equation 2.138 that the motion of each element of the fluid within the surface S is compounded of a motion of translation, a motion of rotation, and a motion of dilatation alike in all directions. So far as regards the first two kinds of motion, the fluid element moves like a solid, and of course there is nothing to call internal friction into play. For the reasons stated in my former paper, I was led to assume that a motion of dilatation alike in all directions, (which of course can only exist in the case of an elastic fluid,) has no effect in causing the pressure to differ from the statical pressure corresponding to the actual density, that is, in occasioning a violation of the functional relation commonly supposed to exist between the pressure, density, and temperature. The reader will observe that this is a totally different thing from assuming that a motion of dilatation has no effect on the pressure at all.

When the fluid is incompressible, then $\delta = 0$, and it may be proved without difficulty that ω', ω'', ω''' are constant, that is to say, constant so far as the co-ordinates are concerned. In this case we get by integrating Equation 2.137

$$\left.\begin{aligned} u = a - \omega'''y + \omega''z\,, \\ v = b - \omega'z + \omega'''x\,, \\ w = c - \omega''x + \omega'y\,. \end{aligned}\right\} \tag{2.139}$$

Hence, in the case of an incompressible fluid, unless the whole mass comprised

within the surface S move together like a solid, there cannot fail to be a certain portion of *vis viva* lost by internal friction. In the case of an elastic fluid, the motion which may take place without causing a loss of *vis viva* in consequence of friction is somewhat more general, and corresponds to velocities $u + \Delta u$, $v + \Delta v$, $w + \Delta w$, where u, v, w are the same as in Equation 2.139, and

$$\Delta u = \delta x + 2 \cdot (\alpha x + \beta y + \gamma z) \cdot x - \alpha \cdot \left(x^2 + y^2 + z^2\right) ,$$

with similar expressions for Δv and Δw. In these expressions α, β, γ are three constants symmetrically related to x, y, z, and δ is a constant which has the same relation to each of the co-ordinates[20].

51. By means of the expression given in Art. 49, page 190, for the loss of *vis viva* due to internal friction, we may readily obtain a very approximate solution of the problem: To determine the rate at which the motion subsides, in consequence of internal friction, in the case of a series of oscillatory waves propagated along the surface of a liquid.
Let the vertical plane of xy be parallel to the plane of motion, and let y be measured vertically downwards from the mean surface; and for simplicity's sake suppose the depth of the fluid very great compared with the length of a wave, and the motion so small that the square of the velocity may be neglected. In the case of motion which we are considering, $u\,\mathrm{d}x + v\,\mathrm{d}y$ is an exact differential $\mathrm{d}\phi$ when friction is neglected, and

$$\phi = c \cdot e^{-my} \cdot \sin(mx - nt) , \tag{2.140}$$

where c, m, n are three constants, of which the last two are connected by a relation which it is not necessary to write down. We may continue to employ this equation as a near approximation when friction is taken into account, provided we suppose c, instead of being constant, to be a parameter which varies slowly with the time. Let V be the *vis viva* of a given portion of the fluid at the end of the time t, then

$$V = \rho c^2 m^2 \cdot \iiint e^{-2my}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z . \tag{2.141}$$

But by means of the expression given in Art. 49, page 190 we get for the loss of *vis viva* during the time $\mathrm{d}t$, observing that in the present case μ is constant, $w = 0$, $\delta = 0$, and $u\,\mathrm{d}x + v\,\mathrm{d}y = \mathrm{d}\phi$, where ϕ is independent of z,

$$4\mu\,\mathrm{d}t \cdot \iiint \left\{ \left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)^2 + \left(\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}\right)^2 + 2 \cdot \left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x\,\mathrm{d}y}\right)^2 \right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z ,$$

[20] see Appendix C at the end.

which becomes, on substituting for ϕ its value,

$$8\mu c^2 m^4 \,\mathrm{d}t \cdot \iiint e^{-2my} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \,.$$

But we get from Equation 2.141 for the decrement of *vis viva* of the same mass arising from the variation of the parameter c

$$-2\rho m^2 c \cdot \frac{\mathrm{d}c}{\mathrm{d}t} \,\mathrm{d}t \cdot \iiint e^{-2my} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \,.$$

Equating the two expressions for the decrement of *vis viva*, putting for m its value $\frac{2\pi}{\lambda}$, where λ is the length of a wave, replacing μ by $\mu'\rho$, integrating, and supposing c_0 to be the initial value of c, we get

$$c = c_0 \cdot e^{-\frac{16\pi^2\mu' t}{\lambda^2}} \,.$$

It will presently appear that the value of $\sqrt{\mu'}$ for water is about 0.0564, an inch and a second being the units of space and time. Suppose first that λ is two inches, and t ten seconds. Then $\frac{16\pi^2\mu' t}{\lambda^2} = 1.256$ and $c : c_0 :: 1 : 0.2848$, so that the height of the waves, which varies as c, is only about a quarter of what it was. Accordingly, the ripples excited on a small pool by a puff of wind rapidly subside when the exciting cause ceases to act.
Now suppose that λ is 40 fathoms or 2880 inches, and that t is 86400 seconds or a whole day. In this case $\frac{16\pi^2\mu' t}{\lambda^2}$ is equal to only 0.005232, so that by the end of an entire day, in which time waves of this length would travel 574 English miles, the height would be diminished by little more than the one two hundredth part in consequence of friction. Accordingly, the long swells of the ocean are but little allayed by friction, and at last break on some shore situated at the distance of perhaps hundreds of miles from the region where they were first excited.

52. It is worthy of remark, that in the case of a homogeneous incompressible fluid, whenever $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, not only are the ordinary equations of fluid motion satisfied[21], but the equations obtained when friction is taken into account are satisfied likewise. It is only the equations of condition which belong to the boundaries of the fluid that are violated. Hence any kind of motion which is possible according to the ordinary equations, and which is such that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, is possible likewise when friction is taken into account, provided we suppose a certain system of normal and tangential pressures to act at the boundaries of the fluid, so as to satisfy the equations of condition. The requisite system of pressures is given by the system of Equation 2.133. Since μ disappears from the general Equation 2.1, it follows that

[21] It is here supposed that the forces X, Y, Z are such that $X\,\mathrm{d}x + Y\,\mathrm{d}y + Z\,\mathrm{d}z$ is an exact differential.

p is the same function as before. But in the first case the system of pressures at the surface was $P_1 = P_2 = P_3 = p$, $T_1 = T_2 = T_3 = 0$. Hence if ΔP_1 &c. be the additional pressures arising from friction, we get from Equation 2.133, observing that $\delta = 0$, and that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential $\mathrm{d}\phi$,

$$\left.\begin{aligned} \Delta P_1 &= -2\mu \cdot \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} \\ \Delta P_2 &= -2\mu \cdot \frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} \\ \Delta P_3 &= -2\mu \cdot \frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} \end{aligned}\right\} \tag{2.142}$$

$$\left.\begin{aligned} \Delta T_1 &= -2\mu \cdot \frac{\mathrm{d}^2\phi}{\mathrm{d}y\,\mathrm{d}z} \\ \Delta T_2 &= -2\mu \cdot \frac{\mathrm{d}^2\phi}{\mathrm{d}z\,\mathrm{d}x} \\ \Delta T_3 &= -2\mu \cdot \frac{\mathrm{d}^2\phi}{\mathrm{d}x\,\mathrm{d}y} \end{aligned}\right\} \tag{2.143}$$

Let $\mathrm{d}S$ be an element of the bounding surface, l', m', n' the direction-cosines of the normal drawn outwards, ΔP, ΔQ, ΔR the components in the direction of x, y, z of the additional pressure on a plane in the direction of $\mathrm{d}S$. Then by Equation 2.9 of my former paper applied to the Equation 2.142, Equation 2.143 we get

$$\Delta P = -2\mu \cdot \left\{ l' \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + m' \frac{\mathrm{d}^2\phi}{\mathrm{d}x\,\mathrm{d}y} + n' \frac{\mathrm{d}^2\phi}{\mathrm{d}x\,\mathrm{d}z} \right\} \tag{2.144}$$

with similar expressions for ΔQ and ΔR, and ΔP, ΔQ, ΔR are the components of the pressure which must be applied at the surface, in order to preserve the original motion unaltered by friction.

53. Let us apply this method to the case of oscillatory waves, considered in Art. 51., page 194. In this case the bounding surface is nearly horizontal, and its vertical ordinates are very small, and since the squares of small quantities are neglected, we may suppose the surface to coincide with the plane of xz in calculating the system of pressures which must be supplied, in order to keep up the motion. Moreover, since the motion is symmetrical with respect to the plane of xy, there will be no tangential pressure in the direction of z, so that the only pressures we have to calculate are ΔP_2 and ΔT_3. We get from Equation 2.140, Equation 2.142, and

Equation 2.143, putting $y = 0$ after differentiation,

$$\left.\begin{aligned} \Delta P_2 &= -2\mu m^2 c \cdot \sin(mx - nt)\,, \\ \Delta T_3 &= -2\mu m^2 c \cdot \cos(mx - nt)\,. \end{aligned}\right\} \tag{2.145}$$

If u_1, v_1 be the velocities at the surface, we get from Equation 2.140, putting $y = 0$ after differentiation,

$$\left.\begin{aligned} u_1 &= mc \cdot \cos(mn - nt)\,, \\ v_1 &= -mc \cdot \sin(mx - nt)\,. \end{aligned}\right\} \tag{2.146}$$

It appears from Equation 2.145 and Equation 2.146 that the oblique pressure which must be supplied at the surface in order to keep up the motion is constant in magnitude, and always acts in the direction in which the particles are moving.
The work of this pressure during the time $\mathrm{d}t$ corresponding to the element of surface $\mathrm{d}x\,\mathrm{d}z$, is equal to $\mathrm{d}x\,\mathrm{d}z\,(\Delta T_3 \cdot u_1\,\mathrm{d}t + \Delta P_1 \cdot v_1\,\mathrm{d}t)$. Hence the work exerted over a given portion of the surface is equal to

$$2\mu m^3 c^2\,\mathrm{d}t \cdot \iint \mathrm{d}x\,\mathrm{d}z\,.$$

In the absence of pressures ΔP_2, ΔT_3 at the surface, this work must be supplied at the expense of *vis viva*. Hence

$$4\mu m^3 c^2\,\mathrm{d}t \cdot \iint \mathrm{d}x\,\mathrm{d}z$$

is the *vis viva* lost by friction, which agrees with the expression obtained in Art. 51, page 194 as will be seen on performing in the latter the integration with respect to y, the limits being $y = 0$ to $y = \infty$.

3. Comparison of Theory and Experiment

3.1. Discussion of the Experiments of Baily, Bessel, Coulomb, and Dubuat

54. The experiments discussed in this Section will be taken in the order which is most convenient for discussion, which happens to be almost exactly the reverse of the chronological order. I commence with the experiments of the late Mr Baily, which are described in the *Philosophical Transactions* for 1832, in a memoir entitled "On the Correction of a Pendulum for the Reduction to a Vacuum: together with Remarks on some anomalies observed in Pendulum experiments."[1]

The object of these experiments was, to determine by actual observation the correction to the time of vibration due to the presence of the air in the case of a great number of pendulums of various forms. This was effected by placing each pendulum in succession in a vacuum apparatus, by which means the pendulum, without being dismounted, could be swung alternately under the full atmospheric pressure, and in air so highly rarefied as nearly to approach to a vacuum. The paper, as originally presented to the Royal Society, contained the results obtained with 41 pendulums, the same body being counted as a different pendulum when swung in a different manner. Out of these, 14 are of such forms as to admit of comparison with theory. An addition to the paper contains the results obtained with 45 pendulums more, of which 24 admit of comparison with theory. The details of these additional experiments are omitted, the results only being given.

Baily has exhibited the results obtained with the several pendulums in each of two ways, first, by the value of the factor n by which the correction for buoyancy must be multiplied in order to amount to the whole effect of the air as given by observation, and, secondly, by the weight of air which must be conceived to be attached to the centre of gyration of the pendulum, adding to its inertia without adding to its weight, in order that the increased inertia, combined with the buoyancy of the air, may account for the whole effect observed. I shall uniformly write $\mathfrak{n}$ for Baily's n, in order to distinguish it from the n of chapter 2 of the present paper, which has a totally different meaning. In the case of a pendulum oscillating in air, it will be sufficient, unless the pendulum be composed of ex-

[1] https://royalsocietypublishing.org/doi/10.1098/rstl.1832.0020

tremely light materials, to add together the effects of buoyancy and inertia. Hence if the pendulum consist of a sphere attached to a fine wire of which the effect is neglected, or else of a uniform cylindrical rod, we may suppose $\mathfrak{n} = 1 + k$, where k is the factor so denoted in chapter 2; so that if M' be the mass of air displaced, kM' will be the mass which we must suppose collected at the centre of the sphere, or distributed uniformly along the axis of the cylinder, in order to express the effect of the inertia of the air. The second mode of exhibiting the effect of the air was suggested by Mr Airy, and is better adapted than the former for investigating the effect of the several pieces of which a pendulum of complicated form is composed. Since the value of the factor $\mathfrak{n}$ and that of the weight of air are merely two different expressions for the result of the same experiment, it would be sufficient to compare either with the result calculated from theory. In some cases, however, I have computed both. In almost all the calculations I have employed 4-figure logarithms. The experimental result is sometimes exhibited to four figures, but no reliance can be placed on the last. In fact, in the best observations, the mean error in different determinations of $\mathfrak{n}$ for the same pendulum appears to have been about the one-hundredth part of the whole, and that it should be so small, is a proof of the extreme care with which the experiments must have been performed.

55. I commence with the 13th set of experiments – *Results with plain cylindrical rods* – page 441. This set contains three pendulums, each consisting of a long rod attached to a knife-edge apparatus. The result obtained with each pendulum furnishes an equation for the determination of μ', and the theory is to be tested by the accordance or discordance of the values so obtained. The principal steps of the calculation are contained in the following Table 3.1.

Pendulum rod	No	Diameter $2a$	Time of vibration τ	$\mathfrak{n}$ by experiment	Correction for confined space (by theory)	Deduced value of k by experiment	Corresponding value of $\mathfrak{m}$	Resulting value of $\sqrt{\mu'}$
Copper, 58.8 inches long	21	0.410	1.0136	2.932	-0.009	1.923	1.5445	0.1166
Brass, 56.4 inches long	43	0.185	0.9933	4.083	-0.002	3.081	0.7000	0.1175
Steel, 56.4 inches long	44	0.072	0.9933	7.530		6.530	0.2822	0.1134

Table 3.1.: Determination of $\sqrt{\mu'}$ by means of Baily's experiments with plain cylindrical rods.

In this table the first column explains itself. The next contains the reference number. In the case of the copper rod I have replaced 42 by 21, under which number the details of the experiment will be found. The diameters of the rods are expressed in decimals of an inch. The time of vibration of the pendulum No. 21 may be got from the tables at the end of BAILY's memoir, which contain the details of the experiments. Nos. 43 and 44 belong to the "additional experiments", of which all the details are suppressed. BAILY has not even given the times of vibration, not having been aware of the circumstance, indicated by the theory of this paper, that the factor $\mathfrak{n}$ and the weight of air which must be conceived as dragged by the pendulum are functions of the time of vibration. Accordingly, in the cases of the pendulums Nos. 43 and 44, and in all similar cases, I have calculated the time of vibration by the ordinary formulæ of dynamics. In calculating τ, I have added 1.55 inch, the length of the shank of the knife-edge apparatus, to the length of the rods. The result so obtained is abundantly accurate enough for my purpose. Had the rod, retaining its actual length, been supposed to begin directly at the knife-edge, the error thence resulting in the value of τ, or rather the correspoding error in the calculatcd value of $\mathfrak{n}$ or k, might just have been sensible. The fifth column in the above table is copied from BAILY's table. The next contains a small correction necessary to reduce the value of $\mathfrak{n}$ got from observation to what would have been got from observations made in an unlimited mass of fluid. It is calculated from the formula $\frac{2a^2}{(b^2-a^2)}$ or $\frac{2a^2}{b^2}$ nearly, which is obtained from the ordinary equations of hydrodynamics, and therefore it cannot be regarded as more than a rude approximation. It will be useful, however, as affording an estimate of the magnitude of the effect produced by confining the air. The diameter of the vacuum tube (whether external or internal is not specified) is stated to have been six inches and a half, whence $2b = 6.5$. The values of k given in the next column are obtained by applying the correction for confined space to BAILY's values of $\mathfrak{n}$, and subtracting unity. The value of $\mathfrak{m}$ corresponding to each value of k was got by interpolation from the table near the end of section 2.3 of the former part of this paper. For $k = 1.923$ the interpolation is easy. The value 3.081 happens to be almost exactly found in the table. For $k = 6.530$, a remark already made will be found to be of importance, namely, that the first differences of $\mathfrak{m}^2 \cdot (k-1)$ are nearly constant. The last column contains the value of $\sqrt{\mu'}$ obtained from the equation

$$\mathfrak{m} = \frac{a}{2} \cdot \sqrt{\frac{\pi}{\mu' \tau}}\,, \tag{3.1}$$

which contains the definition of $\mathfrak{m}$.
It will be observed that the three values of $\sqrt{\mu'}$ are nearly identical. Of course any theory professing to account for a set of experiments by means of a particular value of a disposable constant, when applied to the experiments would lead to nearly the same numerical value of the constant if the experiments were made under nearly the same circumstances. But in the present case the circumstances of the experiments

are widely different. The diameter of the steel rod is little more than the sixth part of that of the copper rod, and the value of k obtained by experiment for the steel rod is more than three times as great as that obtained for the copper rod. It is a simple consequence of the ordinary theory of hydrodynamics that in the case of a long rod oscillating in an unlimited fluid $k = 1$, and we see that this value of k must be multiplied, in round numbers, by 2, by 3, and by $6\frac{1}{2}$, in order to account for the observed effect. The value 1.5445 of $\mathfrak{m}$ is so large that the descending series comes into play in the calculation of the function k, while 0.2822 is so small that the ascending series are rapidly convergent. Hence the near agreement of the values of $\sqrt{\mu'}$ deduced from the three experiments is a striking confirmation of the theory. The mean of the three is 0.1158, but of course the last figure cannot be trusted. I shall accordingly assume as the value of the square root of the index of friction of air in its average state of pressure, temperature, and moisture

$$\sqrt{\mu'} = 0.116\,.$$

It is to be remembered that $\sqrt{\mu'}$ expresses a length divided by the square root of a time, and that the numerical value above given is adapted to an English inch as the unit of length, and a second of mean solar time as the unit of time.[2]

56. I now proceed to compare the observed values of $\mathfrak{n}$ with those calculated from theory with the assumed value of $\sqrt{\mu}$. I begin with the same cylindrical rods as before, together with the long brass tubes Nos. 35 to 38. The diameter of this tube was 1.5 inch, and its length 56 inches. The ends were open, but as the included air was treated by Mr Baily in the reduction of his experiments as if it formed part of the pendulum, we may regard the pendulum as a solid rod. The tube was furnished with six agate planes, represented in the wood-cut at page 417, which rested on fixed knife-edges. The pendulums Nos. 35, 36, 37, and 38 consisted of the same tube swung on the planes marked A, C, a, c. In air the pendulum swung at the rate of about 90080 vibrations in a day, so that $\tau = 0.9596$ nearly. The values of $\mathfrak{n}$ obtained with the end planes A, c were slightly though sensibly greater than the values obtained with the mean planes C, a. I shall suppose the mean of the four values of $\mathfrak{n}$, namely 2.290, to be the result of the experiments. In the following Table 3.2 the difference between the theoretical and experimental values of $\mathfrak{n}$ is exhibited both by decimals and as a fractional part of the former of these values.

[2]According to `https://www.engineeringtoolbox.com/air-absolute-kinematic-viscosity-d_601.html` the value for $\mu' = 0.116^2 \frac{\text{in}^2}{\text{sec}}$ corresponds to a temperature of approximately $-57\,^\circ\text{C}$ for air at 1 atmosphere pressure.

No.	$2a$	$\mathfrak{m}$	k	Add for confined space	Total $\mathfrak{n}$ by theory	$\mathfrak{n}$ by experiment	Difference
35 to 38	1.5	5.849	1.242	0.122	2.364	2.290	-0.074, or $-\frac{1}{31}$
21 or 42	0.410	1.555	1.917	0.009	2.926	2.932	$+0.006$, or $+\frac{1}{489}$
43	0.185	0.7089	3.055	0.002	4.057	4.083	$+0.026$, or $+\frac{1}{157}$
44	0.072	0.2759	6.670		7.670	7.530	-0.140, or $-\frac{1}{54}$

Table 3.2.: Baily's results with a long brass tube and with long cylindrical rods

It will be seen at once how closely the experiments are represented by theory. The largest proportionate difference occurs in the case of the brass tube, and even that is less than one thirtieth. A glance at BAILY's wood-cut at page 417 will shew that the six planes with which the tube was furnished caused the whole figure to deviate sensibly from the cylindrical form. Moreover the resistance experienced by each element of the cylinder has been calculated by supposing the element in question to belong to an infinite cylinder oscillating with the same linear velocity, and the resistance thus determined must be a little too great in the immediate neighbourhood of the ends of the cylinder, where the free motion of the air is less impeded than it would be if the cylinder were prolonged. Lastly, the correction for confined space is calculated according to the ordinary equations of hydrodynamics, and on that account, as well as on account of the abrupt termination of the cylinder, will be only approximate. The small discrepancy between theory and observation, as well as the small difference (amounting to about the 1-83rd of the whole) detected by experiment between the results obtained with the extreme planes and those obtained with the mean planes, may reasonably be attributed to some such causes as those just mentioned. In the case of the steel rod or wire, the difference between theory and observation may be altogether removed by supposing a very small error to have existed in the measurement of the diameter of the rod. Since, as we have seen, the observation is satisfied by $\mathfrak{m} = 0.2822$, and Equation 3.1 gives $a \propto \mathfrak{m}$ when μ' and τ are constant, it is sufficient, in order to satisfy the experiment, to increase the diameter of the rod in the ratio of 0.2759 to 0.2822, or to suppose an error of only 0.0017 inch in defect to have existed in the measurement of the diameter.

57. I proceed next to the experiments on spheres attached to fine wires. The pendulums of this construction comprise four $1\frac{1}{2}$-inch spheres, Nos. 1, 2, 3, and 4; three 2-inch spheres, Nos. 5, 6, and 7; and one 3-inch sphere, No. 66. Nos. 8 and 9 are the same spheres as Nos. 5 and 7 respectively, swung by suspending the wire over a cylinder instead of attaching it to a knife-edge apparatus. As this mode of suspension was not found very satisfactory, and the results are marked by BAILY as doubtful cases, I shall omit the pendulums Nos. 8 and 9, more especially as with reference to the present inquiry they are merely repetitions of Nos. 5 and 7. In the case of a sphere attached to a fine wire of which the effect is neglected, and swung in an unconfined mass of fluid, we have by Equation 2.52

$$k = \frac{1}{2} + \frac{9}{2a} \cdot \sqrt{\frac{\mu'\tau}{2\pi}}\,, \tag{3.2}$$

$2a$ being in this case the diameter of the sphere. Before employing this equation in the comparison of theory and experiment, it will be requisite to consider two corrections, one for the effect of the wire, the other for the effect of the confinement of the air by the sides of the vacuum tube.
I have already remarked at the end of section 2.4, that the application of the

formulæ of section 2.3 to the case of such fine wires as those used in pendulum experiments is not quite safe. Be that as it may, these formulæ will at any rate afford us a good estimate of the probable magnitude of the correction.
Let l be the length, a_1 the radius, V_1 the volume of the wire, V the volume of the sphere, I the moment of inertia of the pendulum, I' that of the air which we may conceive dragged by it, H the sum of the elements of the mass of the pendulum multiplied by their respective vertical distances below the axis of suspension, H' the same for the air displaced, σ the density of the air. Then the length of the isochronous simple pendulum is $\frac{I}{H}$ in vacuum, and $\frac{I+I'}{H-H'}$ in air, and the time of vibration is increased by the air in the ratio of $\frac{\sqrt{I}}{\sqrt{H}}$ to $\frac{\sqrt{I+I'}}{\sqrt{H-H'}}$, or, on account of the smallness of σ, in the ratio of 1 to $I + \frac{1}{2} \cdot \left(\frac{I'}{I} + \frac{H'}{H}\right)$ nearly. Now $\frac{1}{2} \cdot \frac{H'}{H}$ is the correction for buoyancy, and therefore

$$\mathfrak{n} - 1 = \frac{I'}{H'} \cdot \frac{H}{I} \,. \tag{3.3}$$

We have also, if k_1 be the value of the function k of section 2.3,

$$\left.\begin{aligned} I' &= k\sigma V \cdot (l+a)^2 + \frac{1}{3} k_1 \sigma V_1 l^2 \,, \\ H' &= \sigma V \cdot (l+a) + \frac{1}{2} \sigma V_1 l \,, \end{aligned}\right\} \tag{3.4}$$

and $\frac{H}{I} = \frac{1}{(l+a)}$ very nearly. Substituting in Equation 3.3, expanding the denominator, and neglecting V_1^2, we get

$$\mathfrak{n} - 1 = k + \frac{1}{3}\frac{V_1}{V} \cdot k_1 \cdot \left(\frac{l}{(l+a)}\right)^2 - \frac{1}{2}\frac{V_1}{V} \cdot k \cdot \frac{l}{(l+a)} \,.$$

Now V_1 is very small compared with V, and it is only by being multiplied by the large factor k_1 that it becomes important. We may then, without any material error, replace the last term in the above equation by $\frac{1}{3}\frac{V_1}{V} \cdot \frac{l^2}{(l+a)^2}$, and if λ be the length of the isochronous simple pendulum, we may suppose $l + a = \lambda$, and replace $\frac{l^2}{(l+a)^2}$ by $1 - \frac{2a}{\lambda}$, since a is small compared with λ. We thus get, putting $\Delta\,\mathfrak{n}$ for the correction due to the wire,

$$\Delta\,\mathfrak{n} = \frac{1}{3}\frac{V_1}{V} \cdot \left(1 - \frac{2a}{\lambda}\right) \cdot (k_1 - 1) \,.$$

Substituting for $k_1 - 1$ from Equation 2.115, and for $\mathfrak{m}$ from Equation 3.1, in which equations, however, k_1, a_1 must be supposed to be written for k, a, expressing V_1, V in terms of the diameters of the wire and sphere, and neglecting as before a^2 in

comparison with λ^2, we get

$$\Delta\,\mathfrak{n} = \frac{(2\lambda - 3\cdot 2a)\cdot \mu'\tau}{\left\{L^2 + \left(\frac{\pi}{4}\right)^2\right\}\cdot (2a)^3}\,, \tag{3.5}$$

where

$$-L = \ln\left(\frac{4}{2a_1}\right)\cdot\sqrt{\frac{\mu'\tau}{\pi}} - 0.5772\,. \tag{3.6}$$

It is by these formulæ that I have computed the correction for the wire in the following table. In the experiments, the time of oscillation was so nearly one second that it is sufficient in the Equation 3.2, Equation 3.5, and Equation 3.6 to put $\tau = 1$, and take λ for the length of the second pendulum, or 39.14 inches.

With respect to the correction for confined space, it seems evident that the vacuum tube must have impeded the free motion of the air, and consequently increased the resistance experienced by the pendulum when it was swung in air, and that the increase of resistance caused by the cylindrical tube must have been somewhat less than that which would have been produced by a spherical envelope of the same radius surrounding the sphere. The effect of a spherical envelope has been investigated in section 2.2; but as we are obliged at last to have recourse to estimation, it is needless to be very precise in calculating the increase of resistance due to such an envelope, and we may accordingly employ the expression obtained from the ordinary theory of hydrodynamics. According to this theory, the increase of the factor k, which is due to the envelope, is equal to $\frac{3}{2}\cdot a^3\cdot\frac{1}{(b^3-a^3)}$, or $\frac{3}{2}\cdot\frac{a^3}{b^3}$ nearly, when b is large compared with a. The increase due to a cylindrical envelope whose axis is vertical, and consequently perpendicular to the direction of oscillation of the sphere, may be estimated at about two-thirds of the increase due to a spherical envelope of the same diameter. I have accordingly taken $+\frac{a^3}{b^3}$ for the correction for confined space, and have supposed $2b = 6.5$ inches.

The diameter of the wire employed in the pendulums Nos. 1, 2, 3, 5, 6, and 7, is stated to have been about the $\frac{1}{70}$th of an inch, and that of the wire employed with the heavy brass sphere No. 66, about 0.023 inch. The ivory sphere No. 4 was swung with a fine wire weighing rather more than half a grain. Taking the weight at half a grain, and the specific gravity of silver at 10.5, we have for this wire $2a_1 = 0.00251$ nearly. The diameters of the three brass spheres in the following Table 3.3 are taken from page 447 of Baily's memoir. The several parts of which, according to theory, $\mathfrak{n}$ is composed, are exhibited separately.

No. and kind	Diameter of sphere, $2a$	Diameter of wire, $2a_1$	$\mathfrak{n}$ by theory						$\mathfrak{n}$ by experiment	Difference
			For buoyancy	For inertia on common theory	Additional for inertia on account of internal friction	Correction for wire	Correction for confined space (estimated)	Total		
$1\frac{1}{2}$-INCH SPHERES										
No. 1, Platina	1.44	0.014 29	1	0.5	0.289	0.035	0.011	1.835	1.881	$+0.046$, or $+\frac{1}{40}$
No, 2, Lead	1.46	0.014 29	1	0.5	0.285	0.035	0.011	1.831	1.871	$+0.040$, or $+\frac{1}{46}$
No. 3, Brass	1.465	0.014 29	1	0.5	0.284	0.035	0.011	1.830	1.834	$+0.004$, or $+\frac{1}{457}$
No. 4, Ivory	1.46	0.002 51	1	0.5	0.285	0.016	0.011	1.812	1.872	$+0.060$, or $+\frac{1}{30}$
2-INCH SPHERES										
No. 5, Lead	2.06	0.014 29	1	0.5	0.202	0.012	0.032	1.746	1.738	-0.008, or $-\frac{1}{218}$
No. 6, Brass	2.065	0.014 29	1	0.5	0.202	0.012	0.032	1.746	1.751	$+0.005$, or $+\frac{1}{349}$
No. 7, Ivory	2.06	0.014 99	1	0.5	0.202	0.012	0.032	1.746	1.755	$+0.009$, or $+\frac{1}{194}$
3-INCH SPHERE										
No. 66, Brass	3.030	0.023	1	0.5	0.137	0.005	0.101	1.743	1.748	$+0.005$, or $+\frac{1}{349}$

Table 3.3.: Baily's results with spheres suspended by fine wires

The mean error in different determinations of $\mathfrak{n}$ for the same sphere was about 0.01 or 0.02, and this does not include errors arising from small errors in specific gravities, &c. Hence, if we except the spheres Nos. 1, 2, and 4, the discrepancies between theory and experiment are altogether insignificant. In considering the confirmation thence arising to the theory, it must be borne in mind that the theory did not furnish a single disposable constant, inasmuch as $\sqrt{\mu'}$ was already determined from the experiments with cylindrical rods. The result obtained with the brass sphere No. 3 happens to agree almost exactly with theory. However, as the results obtained with this sphere exhibited some anomalies, it seems best to exclude it from consideration, The value of $\mathfrak{n}$, then, which belongs to a $1\frac{1}{2}$-inch sphere, appears to exceed by a minute quantity the value deduced from theory. The difference is indeed so small that it might well be attributed to errors of observation, were it not that all the spheres tell the same tale. Thus the error +0.046 in the case of the platina sphere corresponds to an error of less than the fortieth part of a second in the observation of an interval of time amounting to $4\frac{1}{2}$ hours. If the apparent defect, amounting to about 0.04 or 0.05, in the theoretical result be real, it may be attributed with probability to an error in the correction for the wire. This would be no objection to the theory, for it will be remembered that the theory itself indicated the probable failure of the formulæ generally applicable to a long cylinder when the cylinder comes to be of such extreme fineness as the wires employed in pendulum experiments.

58. The preceding experiments of Baily's are the most important for the purposes of the present paper, inasmuch as they were performed on pendulums of simple and very different forms; but there still remain three sets of experiments, the fourteenth, fifteenth, and sixteenth, in which the pendulum consisted of a combination of a sphere and a rod, so that the results can be compared with theory. The details of these experiments being suppressed, I have been obliged to calculate the time of oscillation from the ordinary formulæ of dynamics, but the results will no doubt be accurate enough for the purpose required. In all the calculations I have supposed the rod to reach up to the axis of suspension, and have consequently added 1.55 inch (the length of the shank of the knife-edge apparatus) to the length of the rod, and have added to the weight of the rod a quantity bearing to the whole weight the ratio of 1.55 inch to the whole length.
In the case of the spheres attached to the ends of the rods (sets 14 and 16) the process of calculation is as follows. Let l be the length of the rod increased by 1.55 inch, W_1 its weight, increased as above explained, a the radius and W the weight of the sphere, λ the length of the isochronous simple pendulum. Then supposing the masses of the rod and sphere to be respectively distributed along the axis, and collected at the centre, which will be quite accurate enough for the present purpose, and putting a for the ratio of a to l, we have by the ordinary formula

$$\lambda = \frac{\frac{1}{3}W_1 + (1+a)^2 \cdot W}{\frac{1}{2}W_1 + (1+a) \cdot W} \cdot l\,, \tag{3.7}$$

whence τ, the time of vibration, is known. Equation 3.2 then gives k, which applies to the sphere, and Equation 3.1 gives $\mathfrak{m}$, the a in this formula being the radius of the rod, from whence k_1, which applies to the rod, may be got by interpolation from Table 2.1. Let Δk, Δk_1 be the corrections which must be applied to k, k_1 on account of the confined space of the vacuum apparatus, and let S_1, S be the specific gravities of the rod and sphere respectively; then we get by means of Equation 3.3 and Equation 3.4

$$\mathfrak{n}-1 = \frac{\frac{1}{3}\cdot(k_1+\Delta k_1)\cdot\frac{W_1}{S_1}+(1+a)^2\cdot(k+\Delta k)\cdot\frac{W}{S}}{\frac{1}{3}W_1+(1+a)^2\cdot W}\cdot\frac{\frac{1}{2}W_1+(1+a)\cdot W}{\frac{1}{2}\cdot\frac{W_1}{S_1}+(1+a)\cdot\frac{W}{S}}\,. \quad (3.8)$$

The first of the two factors connected by the sign $\cdot$ in this equation is equal to $\frac{I'}{\sigma\cdot I}$, and if we want to calculate the weight of air which we must conceive attached to the centre of gyration of the pendulum in order to allow for the inertia of the air, we have only to multiply the factor just mentioned by σ and by the weight of the whole pendulum. The following table contains the comparison of theory and experiment in the case of the 14th set. The rods here mentioned are the same as those which composed the pendulums Nos. 21, 43, and 44, and the spheres are the three brass spheres of Nos. 3, 5, and 66. It appears from p. 432 of Baily's paper that his results are all reduced to a standard pressure and temperature, on the supposition that the effect of the air on the time of vibration is proportional to its density. The theory of the present paper shews that this will only be the case if μ' be constant, which however there is reason for supposing it to be when the pressure alone varies. Be that as it may, no material error can be produced by reducing the observations in this way, because the difference of density in any pair of experiments did not much differ from the density of air at the standard pressure and temperature. The standard pressure and temperature taken were 29.9218 inches of mercury and 32 °F, and the assumed specific gravity of air at this pressure and temperature was the l-770th of that of water, so that in the calculations from theory it is to be supposed that $\frac{1}{\sigma}=770$.

If w be the weight of the whole pendulum, w' that of the air which we must suppose attached to the pendulum at its centre of gyration in order to express the effect of the inertia of the air, S the *vibrating* specific gravity of the pendulum, the effects of buoyancy and inertia are as $\frac{\sigma}{S}$ to $\frac{w'}{w}$; but they are also as 1 to $(\mathfrak{n}-1)$, according to the definition of the factor $\mathfrak{n}$, and therefore

$$w' = (\mathfrak{n}-1)\cdot\frac{\sigma}{S}\cdot w\,, \quad (3.9)$$

a formula which may be employed to calculate w' when $\mathfrak{n}$ is known.

No. and nature of pendulum	Value of $\mathfrak{n}$			Weight of adhesive air, in grains		
	By theory	By experiment	Difference	By theory	By experiment	Difference
No. 45 – $1\frac{1}{2}$-inch sphere with copper rod	2.525	2.458	-0.067, or $-\frac{1}{38}$	4.863	4.564	-0.299, or $-\frac{1}{16}$
No, 46 – 2-inch sphere with ditto	2.202	2.234	$+0.032$, or $+\frac{1}{69}$	5.005	5.076	$+0.071$, or $+\frac{1}{70}$
No. 47 – 3-inch sphere with ditto	1.957	1.873	-0.084, or $-\frac{1}{23}$	7.071	6.425	-0.646, or $-\frac{1}{11}$
No. 48 – $1\frac{1}{2}$-inch sphere with brass rod	2.375	2.356	-0.019, or $-\frac{1}{125}$	1.447	1.417	-0.030, or $-\frac{1}{48}$
No. 49 – 2-inch sphere with ditto	2.060	1.982	-0.078, or $-\frac{1}{26}$	2.135	1.973	-0.162, or $-\frac{1}{13}$
No. 50 – 3-inch sphere with ditto	1.631	1.933 ?	$+0.302$?	4.411	4.868 ?	$+0.457$?
No. 51 – $1\frac{1}{2}$-inch sphere with steel rod	2.099	2.344 ?	$+0.245$?	0.682	0.834 ?	$+0.152$?
No. 52 – 2-inch sphere with ditto	1.920	1.793	-0.127, or $-\frac{1}{15}$	1.457	1.259	-0.198, or $-\frac{1}{7}$
No. 53 – 3-inch sphere with ditto	1.781	1.759	-0.022, or $-\frac{1}{81}$	3.742	3.670	-0.072, or $-\frac{1}{52}$

Table 3.4.: Baily's results with spheres at the ends of long rods

With respect to the two experiments marked ? BAILY remarks, "These two experiments (with the pendulums Nos. 50 and 51) are very unsatisfactory; and are marked as such in my journal. It was consequently my intention to have repeated them: but the subject was overlooked till it was too late. I should propose their being rejected altogether." If these two experiments be struck out, it will be seen that the differences between theory and experiment are very small, especially when the difficulty of this set of experiments is considered, arising from the frequency of the coincidences with the mean solar clock.

59. On account of the difficulty which BAILY experienced in obtaining accurate results with the long rods and spheres attached, he divided the brass and steel rods near the centre of oscillation, and after having cut off an inch from each portion inserted the spheres where the rods had been divided. The results thus obtained constitute the 15th set of experiments. He afterwards removed the lower segments of the rods, and obtained the results contained in the 16th set. I shall give the computation of the latter set first, inasmuch as the formulæ to be employed are exactly the same as those required for the 14th set. The experiments belonging to this set in which the spheres were swung with iron wires have already been computed under the head of spheres attached to fine wires.
Here again (Table 3.5) the differences between theory and experiment are extremely small. In the ease of the pendulum No. 61, BAILY's two results 1.901 and 1.513 appear to be inconsistent, as not agreeing with Equation 3.9.

No. and nature of pendulum	Value of $\mathfrak{n}$			Weight of adhesive air, in grains		
	By theory	By experiment	Difference	By theory	By experiment	Difference
No. 60 – $1\frac{1}{2}$-inch sphere with brass rod	2.149	2.198	$+0.049$ or $+\frac{1}{44}$	1.011	1.047	$+0.036$ or $+\frac{1}{28}$
No. 61 – 2-inch sphere with ditto	1.879	1.901	$+0.022$ or $+\frac{1}{85}$	1.619	1.513	-0.106 or $-\frac{1}{15}$
No. 62 – 3-inch sphere with ditto	1.787	1.830	$+0.043$ or $+\frac{1}{42}$	3.970	4.202	$+0.232$ or $+\frac{1}{17}$
No. 63 – $1\frac{1}{2}$-inch sphere with steel rod	1.960	1.904	-0.056 or $-\frac{1}{35}$	0.570	0.537	-0.033 or $-\frac{1}{16}$
No. 64 – 2-inch sphere with ditto	1.796	1.785	-0.011 or $-\frac{1}{163}$	1.239	1.227	-0.012 or $-\frac{1}{103}$
No. 65 – 3-inch sphere with ditto	1.758	1.779	$+0.021$ or $+\frac{1}{84}$	3.609	3.720	$+0.111$ or $+\frac{1}{32}$

Table 3.5.: Baily's results with the spheres at the end of the short rods

60. The following Table 3.6 contains the values of τ, k and k_1 deduced from the given data, and employed in the calculations of which the results are contained in the two preceding tables. It is added, partly to facilitate a comparison of the circumstances of the different experiments, partly to assist in the re-computation of any of the experiments, or the detection of any numerical error which I may have committed. I may here observe that I have not, generally speaking, re-examined the calculations, except where an error was apparent, but that each step requiring addition, subtraction, multiplication, or division, was checked immediately after it was performed. I have not thought it requisite to check in this manner the taking of logarithms or antilogarithms out of a table.

Sphere	Rod	Long rods				Short rods			
		No.	τ	k	k_1	No.	τ	k	k_1
$1\frac{1}{2}$-inch	copper	45	1.090	0.7968	1.951				
2-inch	copper	46	1.158	0.7170	1.981				
3-inch	copper	47	1.227	0.6523	2.010				
$1\frac{1}{2}$-inch	brass	48	1.155	0.8055	3.222	60	0.9517	0.7772	3.012
2-inch	brass	49	1.198	0.7207	3.264	61	0.9806	0.7005	3.042
3-inch	brass	50	1.222	0.6520	3.288	62	0.9982	0.6373	3.062
$1\frac{1}{2}$-inch	steel	51	1.190	0.8099	7.272	63	0.9868	0.7824	6.649
2-inch	steel	52	1.199	0.7208	7.299	64	0.9954	0.7021	6.679
3-inch	steel	53	1.231	0.6525	7.396	65	1.0030	0.6377	6.714

Table 3.6.: Values of τ, k and k_1 employed in the calculation of the theoretical results employed in the two preceding tables

The corrections for confined space employed are, for the spheres, (Δk), 0.0115, 0.0321, 0.1013; and for the rods, (Δk_1), 0.009, 0.002, 0.000. These corrections are to be added to the values of k, k_1 given in the preceding table before going on with the calculation.

61. In the 14th set of experiments, the weight of adhesive air due to the spheres alone has been computed by Baily by subtracting from the whole weight, as given by observation, the weight due to the rods as given by the 13th set of experiments, taking account of the change of weight corresponding to the change in the position of the centre of gyration, the point at which the air is supposed to be attached. According to theory, this process is not legitimate, inasmuch as the weight dragged by a rod is a function of the time of vibration, which is altered when a sphere is

attached to the end of the rod. But in the 15th set of experiments the spheres did not materially affect the time of vibration, inasmuch as they were inserted nearly at the centre of oscillation of the rods, and therefore in this case the process is legitimate. Accordingly, I think it is a sufficient comparison between theory and experiment in the case of the 15th set, to compare the weights of air due to the spheres alone, as calculated by Baily, with the weights calculated according to the theory of this paper with the assumed value of $\sqrt{\mu'}$. I have exhibited separately the weight corresponding to the correction for confined space, in order to enable the reader to form an estimate of the extent to which the results may be affected by the uncertainty relating to the amount of this correction.

By Theory				By Experiment			
	$1\frac{1}{2}$-inch sphere	2-inch sphere	3-inch sphere		$1\frac{1}{2}$-inch sphere	2-inch sphere	3-inch sphere
In free air	0.431	1.060	3.002	From exp. with brass rod	0.446	1.180	3.382
Additional for confined space	0.006	0.048	0.476	From exp. with steel rod	0.405	1.039	3.371
Total	0.437	1.108	3.478	Mean	0.425	1.109	3.377
Diff. th. & exp., as decimal	−0.012	+0.001	−0.101	Diff., as fraction of the whole	$-\frac{1}{36}$	$+\frac{1}{1108}$	$-\frac{1}{34}$

Table 3.7.: Weights of air dragged by the spheres alone, as deduced from Baily's results with the spheres at the centre of oscillation of the long rods

62. I pass now to BESSEL's experiments described in his memoir entitled *Untersuchungen über die Länge des einfachen Sekundenpendels*[3], which is printed among the memoirs of the Academy of Sciences of Berlin for the year 1826. The object of this memoir was to determine the length of the seconds' pendulum by a new method, which consisted in swinging the same sphere with wires of two different lengths, the difference of lengths being measured with extreme precision. In the calculation, the absolute length of the simple pendulum isochronous with either the long or the short compound pendulum was regarded as unknown, but the difference of the two as known, and this difference, combined with the observed times of oscillation, is sufficient for the determination of the quantity sought. Nothing more would have been required if the pendulums had been swung in a vacuum; but inasmuch as they were swung in air, a further correction was necessary to reduce the observations to a vacuum. Since it is necessary to take into account the inertia of the air, as well as its buoyancy, in reducing the observations to a vacuum, BESSEL sought to determine by experiment the value of the factor k, of which the meaning has been already explained. The value of this factor, as BESSEL remarked, will depend upon the form of the body; but he does not seem, at least in his first memoir, to have contemplated the possibility of its depending on the time of oscillation, and consequently he supposed it to have the same value for the long as for the short pendulum. When the factor k is introduced, the equation obtained from the known difference of length of the two simple pendulums contains two unknown quantities, namely k, and the length of the seconds' pendulum. To obtain a second equation, BESSEL made another set of experiments, in which the brass sphere was replaced by an ivory sphere, having as nearly as possible the same diameter. The results obtained with the ivory sphere furnished a second equation, in which k appeared with a much larger coefficient, on account of the lightness of ivory compared with brass. The two equations determined the two unknown quantities.

Let λ be the length of the seconds' pendulum, t_1, t_2 the times of oscillation of the brass sphere when swung with the short wire and long wire respectively, l_1, l_2 the lengths of the corresponding simple pendulums, corrected for everything except the inertia of the air, m the mass of the sphere, m_1 the mass of the fluid displaced;then

$$\lambda t_1^2 \cdot \frac{1}{\left(1 + \frac{m_1}{m} \cdot k\right)} = l_1 \,;$$

or, since m_1 is so small that we may neglect m_1^2,

$$\lambda t_1^2 \cdot \left(1 - \frac{m_1}{m} \cdot k\right) = l_1 \,.$$

The long pendulum furnishes a similar equation, and the result obtained from the

[3] https://www.digitale-sammlungen.de/de/view/bsb10052717?page=1

brass sphere is

$$\lambda \cdot \left(t_2^2 - t_1^2\right) \cdot \left(1 - \frac{m_1}{m} \cdot k\right) = l_2 - l_1 \,, \tag{3.10}$$

since $l_2 - l_1$ is the quantity which is regarded as accurately known. The ivory sphere in like manner furnishes the equation

$$\lambda \cdot \left(t_2'^2 - t_1'^2\right) \cdot \left(1 - \frac{m_1}{m'} \cdot k\right) = l_2' - l_1' \,, \tag{3.11}$$

where the accented letters refer to that sphere. The equation for the determination of k results from the elimination of λ between Equation 3.10 and Equation 3.11. Now, according to the theory of this paper, the factor k has really different values for the long and short pendulums. Let k_1 refer to the short, and k_2 to the long pendulum with the brass sphere, k_1' to the short, and k_2' to the long pendulum with the ivory sphere. Then

$$\lambda t_1^2 \cdot \left(1 - \frac{m_1}{m} \cdot k_1\right) = l_1 \,,$$
$$\lambda t_2^2 \cdot \left(1 - \frac{m_1}{m} \cdot k_2\right) = l_2 \,,$$

and therefore

$$l_2 - l_1 = \lambda t_2^2 \cdot \left(1 - \frac{m_1}{m} \cdot k_2\right) - \lambda t_1^2 \cdot \left(1 - \frac{m_1}{m} \cdot k_1\right) \,. \tag{3.12}$$

In the equation resulting from the elimination of λ between Equation 3.10 and Equation 3.11, let the values of $l_2 - l_1$ and $l_2' - l_1'$ got from Equation 3.12 and the similar equation relating to the ivory sphere be substituted. The result is

$$\begin{aligned}&\left(t_2^2 - t_1^2\right) \cdot \left(1 - \frac{m_1}{m} \cdot k\right) \cdot \left\{t_2'^2 \cdot \left(1 - \frac{m_1}{m'} \cdot k_2'\right) - t_1'^2 \cdot \left(1 - \frac{m_1}{m'} \cdot k_1'\right)\right\} \\ &= \left(t_2'^2 - t_1'^2\right) \cdot \left(1 - \frac{m_1}{m'} \cdot k\right) \cdot \left\{t_2^2 \cdot \left(1 - \frac{m_1}{m} \cdot k_2\right) - t_1^2 \cdot \left(1 - \frac{m_1}{m} \cdot k_1\right)\right\} \,.\end{aligned}$$

This equation is of the form

$$P + Qm_1 + Rm_1^2 = P' + Q'm_1 + R'm_1^2 \,,$$

and $P = P'$, and Rm_1^2, $R'm_1^2$ may be neglected, so that the equation is reduced to

$$Q = Q' \,.$$

It is now no longer necessary to distinguish between t_2 and t_2', and between t_1 and t_1', which may be supposed equal. Also $m : m' :: S : S'$, where S, S' are the specific gravities of the brass and ivory spheres respectively. Substituting in the equation

$Q = Q'$, and solving with respect to k, we get

$$k = \frac{t_2^2 \cdot (Sk_2' - S'k_2) - t_1^2 \cdot (Sk_1' - S'k_1)}{(t_2^2 - t_1^2) \cdot (S - S')} \,. \tag{3.13}$$

This equation contains the algebraical definition of that function k of which the numerical value is determined by combining, in Bessel's manner, the results obtained with the four pendulums. Since the equation is linear so far as regards k, k_1 &c., we may consider separately the different parts of which these quantities are composed, and add the results. For the part which relates to the spheres, regarded as suspended by infinitely fine wires, we have $k_2' = k_2$ and $k_1' = k_1$, since the radii of the two spheres were equal, or at least so nearly equal that the difference is insensible in the present enquiry. We get then from Equation 3.13

$$k = \frac{t_2^2 k_2 - t_1^2 k_1}{(t_2^2 - t_1^2)} \,, \tag{3.14}$$

which gives

$$\frac{k - k_1}{t_2^2} = \frac{k - k_2}{t_1^2} = \frac{k_2 - k_1}{(t_2^2 - t_1^2)} \tag{3.15}$$

Since $t_2 > t_1$ and $k_2 > k_1$, the Equation 3.15 shew that the value of k determined by Bessel's method is greater than the factor which relates to the short pendulum, which was a seconds' pendulum nearly, and even greater than that which relates to the long pendulum, as has been already remarked in Art. 6., page 140.
If k_s be the factor relating to either sphere oscillating once in a second, and if the effect of the confinement of the air be neglected, we have from Equation 3.2

$$k_1 - \frac{1}{2} : k_2 - \frac{1}{2} : k_s - \frac{1}{2} :: \sqrt{t_1} : \sqrt{t_2} : 1 \,,$$

and in Bessel's experiments $t_1 = 1.001$, $t_2 = 1.721$, $2a = 2.143$ in English inches. We thus get from either of Equation 3.14 or Equation 3.15, on substituting 0.116 for $\sqrt{\mu'}$, $k = 0.786$. The value of the factor k_s, which relates to a sphere of the same size, swung as a seconds' pendulum, is only 0.694, and k_1 may be regarded as equal to k_s. Equation 3.2 gives $k_2 = 0.755$.

63. We have next to investigate the correction for the wire. The effect of the inertia of the air set in motion by the wire was altogether neglected by Bessel, and indeed it would have been quite insensible had the parts of the correction for inertia due to the wire and to the sphere, respectively, been to each other in nearly the same ratio as the parts of the correction for buoyancy. Baily, however, was led to conclude from his experiments that the effect of the wire was probably not altogether insignificant, and the theory of this paper leads, as we have seen, to the result that the factor $\mathfrak{n}$ is very large in the case of a very fine wire.

The ivory sphere in BESSEL's experiments was swung with a finer wire than the brass sphere. It was for this reason that I did not from the first suppose $k'_1 = k_1$ and $k'_2 = k_2$. Let Δk, Δk_1, &c. be the corrections due to the wire. The values of Δk_1, Δk_2, $\Delta k'_1$, $\Delta k'_2$, may be got from Equation 3.5, in which it is to be remembered that λ denotes the length of the isochronous simple pendulum, not, as in BESSEL's notation, the length of the seconds' pendulum. It is stated by BESSEL (p. 131), that the wire used with the brass sphere weighed 10.95 Prussian grains in the case of the long pendulum, and 3.58 grains in the case of the short. This gives 7.37 grains for the weight of one toise or 72 French inches. The weight of one toise of the wire employed with the ivory sphere was 6.28 – 2.04 or 4.24 grains (p. 141). The specific gravity of the wire was 7.6 (p. 40), and the weight of a cubic line (French) of water is about 0.1885 grain. From these data it results that the radii of the wires were 0.003867 and 0.002933 inch English. Equation 3.1 gives $\mathfrak{m}$, whence L is known from Equation 3.6. The lengths of the isochronous simple pendulums were about 39.20 inches for the short pendulum, and 116.94 for the long. On substituting the numerical values we get from Equation 3.5, since $k_1 = (\mathfrak{n}_1 - 1)$ and $k_2 = (\mathfrak{n}_2 - 1)$,

$$\Delta k_1 = 0.0107\,, \qquad \Delta k'_1 = 0.0090\,,$$
$$\Delta k_2 = 0.0286\,, \qquad \Delta k'_2 = 0.0244\,.$$

The specific gravities of the two spheres were about 8.190 and 1.794, whence we get from Equation 3.13 $\Delta k = 0.0308$, or 0.031 nearly.
The value of k deduced by BESSEL from his experiments was 0.9459 or 0.946 nearly, which in a subsequent paper he increased to 0.956. In this paper he contemplates the possibility of its being different in the cases of the long and of the short pendulum, and remarks with justice that no sensible error would thence result in the length of the seconds' pendulum, as determined by his method, but that the factor k would belong to the system of the two pendulums.
The following is the result of the comparison of theory and experiment in the case of BESSEL's experiments on the oscillations of spheres in air.

Value of k belonging to the system of a long and a short pendulum, as determined experimentally by Bessel	0.956
Value deduced from theory, including the correction for the wire, but not the correction for confined space	0.817
difference	+0.139

I cannot find that BESSEL has stated exactly the distance of the centre of the sphere from the back of the frame within which it was swung, but if we may judge by the sketch of the whole apparatus which is given in Plate I., and by a comparison of figs. 2 and 3, Plate II., it must have been very small, that is to say, a small

fraction of the radius of the sphere[4]. If so, although the exact calculation of the correction for confined space would form a problem of extreme difficulty, it may be shewn from theoretical considerations that the correction would be by no means insensible, so that it might wholly or in part account for the difference +0.139 between the results of theory and observation. It is, however, not improbable, for a reason which has been already mentioned, that the theoretical correction for the wire is not quite exact.

64. The experiments performed by BESSEL on a sphere vibrating in water will be more conveniently considered after the discussion of some experiments of COULOMB's, to which I now proceed. These experiments are contained in a memoir entitled *Expériences destinées à déterminer la cohérence des fluides et les lois de leur résistance dans les mouvements trés-lents*[5], which will be found in the 3rd Volume of the *Mémoires de l'Institut*, p. 246. The experiments which I shall first consider are those which relate to the oscillations of disks suspended in water with their planes horizontal. In these experiments the disk operated upon was attached to the lower extremity of a vertical cylinder of copper, not quite half an inch in diameter, the axis of which passed through the centre of the disk. The cylinder was suspended by a fine wire attached to its upper extremity. The under portion of the cylinder, together with the attached disk, were immersed in water, the disk at the bottom of the cylinder being immersed to the depth of 4 or 5 centimetres below the surface. The upper portion carried a horizontal metallic graduated disk, by means of which the arc of oscillation could be read off, and which, on account of its size and weight, mainly determined the inertia of the system, so that the time of oscillation in the different experiments was nearly the same. The observations were taken as follows. The whole system was turned very slowly round by applying the hands of the graduated disk, taking care not to derange the vertical position of the suspending wire. The arc through which the system had been turned was read by means of the graduation, or rather the system was turned through an arc previously fixed on; the system was then left to itself, and the arc again read off to a certain number of oscillations. Thus it was the decrement of the arc of oscillation that was observed; the time of oscillation was indeed also observed, but only approximately, for the sake of determining a subsidiary quantity required in the calculation. Indeed, it will be easily seen that the experiments were not adapted to determine the effect of the fluid on the time of oscillation. The decrement of arc so determined had to be corrected for the effect of the imperfect elasticity of

[4]The measurement of either of BESSEL's figures, figs. 5 or 6, Plate II. gives 1.53 inch for the distance of the centre of the sphere from the surface of the broad iron bar which formed the back of the frame, the surface of the bar being supposed truly vertical; and the measurement of fig. 2 giving 2.06 inches for the diameter of the sphere, it appears that the distance of the surface of the sphere from the surface of the bar was barely equal to half the radius of the sphere.

[5]`https://www.academie-sciences.fr/pdf/dossiers/Coulomb/Coulomb_pdf/Mem1800_p246.pdf`

the wire, and of the resistance of the air against the graduated disk, and of the water against the portion of the copper cylinder immersed. The amount of the correction was determined by repeating the observation when the lower disk had been removed.

It appeared from the experiments, *first*, that with the same disk immersed, the successive amplitudes of oscillation decreased in geometric progression; *secondly*, that with different disks the moment of the resisting force was proportional to the fourth power of the radius. From these laws COULOMB concluded that each small element of any one of the disks experienced a resistance varying as the area of the element multiplied by its linear velocity. It should be observed that COULOMB was only authorized by his experiments to assert this law to be true in the case of oscillations of given period, inasmuch as the time of oscillation was nearly the same in all the experiments.

Let a be the radius of the disk in the fluid, τ the time of oscillation, θ the angular displacement of the disk, measured from its mean position, I the moment of inertia of the whole system; and let $1 : (1 - m)$ be the ratio in which the arc of oscillation is diminished in one oscillation. According to Equation 2.15 we have

$$e^{-n\beta t}$$

for the factor which expresses the ratio of the arc of oscillation at the end of the time t to the initial arc. At the end of one oscillation $t = \tau$, and the value of the above factor is $1 - m$, which is given by observation. Putting for β its value, in which $M\gamma^2 = I$, and $n\tau = \pi$, we get

$$\ln(1 - m) = -\frac{\rho a^4}{I} \cdot \sqrt{\frac{\pi^3 \mu' \tau}{8}}\,. \tag{3.16}$$

Let T be the time of oscillation, and I_0 the moment of inertia, when the under disk is removed: then $I = I_0 \cdot \frac{\tau^2}{T^2}$. Also if M be the mass and R the radius of the large graduated disk, we have $I_0 = \frac{1}{2}MR^2$, neglecting, as COULOMB did, the rotatory inertia of the copper cylinder. Substituting in Equation 3.16, we get

$$\ln \frac{1}{(1 - m)} = \frac{1}{\sqrt{2}} \pi^{\frac{3}{2}} \rho \sqrt{\mu'} \tau^{-\frac{3}{2}} T^2 a^4 \frac{1}{R^2 M}\,. \tag{3.17}$$

Let W be the weight of the disk in grammes. Then the mass of the disk is equal to that of W cubic centimetres or $1000W$ cubic millimetres of water. Hence $M = 1000\rho W$, a millimetre being the unit of length. Substituting in Equation 3.17, and solving with respect to $\sqrt{\mu'}$, we get

$$\sqrt{\mu'} = 1000 \cdot \sqrt{2} \ln 10 \cdot \pi^{-\frac{3}{2}} W \frac{R^2}{T^2} \frac{1}{a^4} \tau^{\frac{3}{2}} \log \left[\frac{1}{(1 - m)} \right]\,, \tag{3.18}$$

No.	Diamter of disk $2a$, in millimetres	Time four oscillations 4τ	$\log\left[\frac{1}{(1-m)}\right]$	Resulting value of $\sqrt{\mu'}$ in inches
1	195	97	0.0568	0.055 19
2	140	92	0.021	0.057 16
3	119	91	0.0135	0.054 36
4	0	91	0.0058	

Table 3.8.: Determination of the value of $\sqrt{\mu'}$ for water from COULOMB's experiments on the decrement of the arc of oscillation of disks, oscillating in their own plane by the force of torsion

and the same value of $\sqrt{\mu'}$ ought to result from different experiments.
The weight of the disk is stated to have been 1003 grammes, and its diameter 271 millimetres, and it made 4 oscillations in 91 seconds. Hence $W = 1003$, $R = 135.5$, $T = 22.75$. The last three factors in Equation 3.18 vary from one experiment to another. After making experiments with three disks of different radii attached to the copper cylinder, COULOMB made another set with nothing attached, for the purpose of eliminating the effect of the imperfect elasticity of the wire. Table 3.8 contains the data furnished by experiment, together with the value of $\sqrt{\mu'}$ deduced from the several experiments. The latter is reduced to the decimal of an English inch, by including $\overline{2}.5952$ (the logarithm of the ratio of a millimetre to an inch) in the logarithm of the constant part of the 2nd member of Equation 3.18.

In correcting the results of the first three experiments for the imperfect elasticity of the wire, COULOMB calculated the values of m given by the four experiments, and subtracted the value given by the fourth from each of the others. But it is at the same time easier and more exact to subtract the value of $\log\left[\frac{1}{(1-m)}\right]$ given by the fourth experiment from that given by each of the others. For if

$$-2c \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}\,, \quad -2c' \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

be the moments of two forces, each varying as the velocity, divided by the moment of inertia, the factors by which the initial arc of oscillation must be multiplied to get the arc at the end of the time t, first, when the two forces act together, secondly, when the second force acts alone, are

$$e^{-(c+c')\cdot t}\,, \quad e^{-c'\cdot t}\,,$$

respectively, and that, whether the time t be great or small. Hence if we subtract the logarithm of the second factor from that of the first we shall get the logarithm of the factor due to the action of the first force alone. But if we put each factor under the form $(1-m)$, and subtract the m of the second factor from the m of the first, we shall not get the m due to the first force alone, unless t be small enough to allow of our neglecting the squares of $c \cdot t$ and $c' \cdot t$, or at least the product $c \cdot t$, $c' \cdot t$. In truth, when $t = \tau$, the quantities m are sufficiently small to be treated in COULOMB's manner without any material error, since the corrected values of $\log(1-m)$, obtained in the two ways, would only differ in the 4th place of decimals. The numbers given in the last column of the above table were calculated from Equation 3.18, on substituting for $\log\left[\frac{1}{(1-m)}\right]$ the numbers found in the first three lines of the 4th column, corrected by subtracting 0.0058. The mean of the three results is 0.05557, but the three experiments are not equally valuable for the determination of $\sqrt{\mu'}$. For the three numbers from which $\sqrt{\mu'}$ was deduced are 0.0510, 0.0152, 0.0077, and a given error in the first of these numbers would produce a smaller error in $\sqrt{\mu'}$ than that which would be produced by the same error in the second, still more, than that which would be produced by the same error in the third. If we multiply the three values of $\sqrt{\mu'}$ by 510, 152, and 77, respectively, and divide the sum of the products by $510 + 152 + 77$ or 739, we get 0.05551. We may then take 0.555 as the result of the experiments. Assuming $\sqrt{\mu'} = 0.0555$ we have

$\log\left[\frac{1}{(1-m)}\right]$ from experiment	0.0568 in No. 1	0.021 in No. 2,	0.0135 in No. 3,
$\log\left[\frac{1}{(1-m)}\right]$ from theory	0.0571	0.0206	0.0137
difference	−0.0003	+0.0004	−0.0002

65. So far the accordance of the theoretical and observed results is no very searching test of the truth of the theory. For, in fact, the theory is involved in the result only so far as this, that it shews that the resistance experienced by a given small element of a disk oscillating in a given period varies as the linear velocity; since the difference of periods in COULOMB's experiments was so small that the effects thence arising would be mixed up with errors of observation. This law is so simple that it might very well result from theories differing in some essential particulars from the theory of this paper. But should the numerical value of $\sqrt{\mu'}$ determined by COULOMB's experiments on disks be found to give results in accordance with theory in totally different cases, then the theory will receive a striking confirmation. Before proceeding to the discussion of other experiments, there are one or two minute corrections to be applied to the value of $\sqrt{\mu'}$ given above, which it will be convenient to consider.

In the first place, the result obtained in Art. 8, page 143 is only approximate, the approximation depending upon the circumstance that the diameter of the revolving

body is large compared with a certain line determined by the values of $\sqrt{\mu'}$ and τ. In the particular case in which the revolving solid is a circular disk, it happens that the approximate solution satisfies the general equations exactly, except so far as relates to the abrupt termination of the disk at its edge[6]. In consequence of this abrupt termination, the fluid annuli in the immediate neighbourhood of the edge are more retarded by the action of the surrounding fluid than they would have been were the disk continued, and consequently the resistance experienced by the disk in the immediate neighbourhood of its edge is actually a little greater than that given by the formula. I have not investigated the correction due to this cause, but it would doubtless be very small.

In the second place, Equation 2.15 is adapted to an indefinite succession of oscillations, whereas COULOMB did not turn the disk through an angle greater than the largest intended to be observed, and suffer one or two oscillations to pass before the observation commenced, but took for the initial arc that at which the disk had been set by the hand. Probably the disk was held in this position for a short time, so that the fluid came nearly to rest. If so, the resulting value of $\sqrt{\mu'}$ as may readily be shewn, would be a little too small. For in the course of an indefinite series of oscillations, the disk, in its forward motion, carries a certain quantity of fluid with it, and this fluid, in consequence of its inertia, tends to preserve its motion. Hence, when the disk, having attained its maximum displacement in the positive direction, begins to return, it finds the fluid moving in such a manner as to oppose its return, and therefore it experiences a greater resistance than if it had started from the same position with the fluid at rest. In fact, it appears from the expression for G in Art. 8, page 143 that the moment of the resistance vanishes, in passing from negative to positive, not when the disk has reached the end of its excursion in the positive direction, but the eighth part of a period earlier. Hence, had the observation commenced during a series of oscillations, a larger initial arc would have been necessary, to overcome the greater resistance, in order to produce, after a given number of oscillations, the same final arc as that actually observed. I have investigated the correction to be applied on account of this cause, and find it to be about +0.009, but I must refer to a note for the demonstration, in order not to interrupt the present discussion[7]. I shall assume then, in the following comparisons, that for water[8]

$$\sqrt{\mu'} = 0.0564\,,$$

the units being the same as before, namely, an English inch and a second. That μ' is independent of the pressure of the fluid, or at least very nearly so, appears from an experiment of COULOMB's, in which it was found that the decrement of the arc of oscillation of a disk oscillating in water was the same in an exhausted receiver as under the full atmospheric pressure. I will here mention another experiment of

[6] See Appendix A at the end.
[7] See Appendix B at the end.
[8] 89

COULOMB's which bears directly on one part of the theory. On covering the disk with a thin coating of tallow, the resistance was found to be the same as before; and even when the tallow was sprinkled with powdered sandstone, by means of a sieve, the increase of resistance was barely sensible. This strikingly confirms the correctness of the equations of condition assumed to hold good at the surface of a solid.

66. I will now compare Equation 3.2 with the results obtained by BESSEL for the oscillations of the brass sphere in water, which will be found at page 65 of his memoir. This sphere was suspended so as to be immersed in the water contained in a large vessel, and was swung with two different lengths of wire, the same as those employed for the experiments in air. The times of oscillation were 1.9085 second for the long pendulum, and 1.1078 for the short. The results are

	Long pendulum	Short pendulum
k, by experiment	0.648	0.602
k, by theory	0.631	0.600
difference	+0.017	+0.002

The depth to which the spheres were immersed is not stated, but it was probably sufficient to render the effect of the free surface small, if not insensible. The vessel was three feet in diameter, and the water 10 inches deep, so that unless the spheres were suspended near the bottom, which is not likely to have been the case, the effect of the limitation of the fluid by the sides of the vessel must have been but trifling. The agreement of theory and observation, as will be seen, is very close.

67. In the same memoir which contains the experiments on disks, COULOMB has given the results of some experiments in which the disk immersed in the fluid was replaced by a long narrow cylinder, placed with its axis horizontal and its middle point in the prolongation of the axis of the vertical copper cylinder. In these experiments, the arcs did not decrease in geometric progression, as would have been the case if the resistance had varied as the velocity; but it was found that the results of observation could be satisfied by supposing the resistance to vary partly as the first power, and partly as the square of the velocity. In COULOMB's notation, $1 : (1 - m)$ denotes the ratio in which the arc of oscillation would be altered after one oscillation, if the part of the resistance varying as the square of the velocity were destroyed. The several experiments performed with the same cylinder were found to be sufficiently satisfied by the formula deduced from the above-mentioned hypothesis respecting the resistance, when suitable numerical values were assigned to two disposable constants m and p, of which p is related to the part of the resistance varying as the square of the velocity.
Conceive the cylinder divided into elementary slices by planes perpendicular to its

axis. Let r be the distance of any slice from the middle point, θ the angle between the actual and the mean positions of the axis, $\mathrm{d}\mathscr{F}$ that part of the resistance experienced by the slice which varies as the first power of the velocity. Then calculating the resistance as if the element in question belonged to an infinite cylinder moving with the same linear velocity, we have by the formulæ of Art. 31., page 169

$$\mathrm{d}\mathscr{F} = k' M' n \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t}\,, \quad \text{where} \quad M' = \pi\rho a^2\,\mathrm{d}r\,,$$
$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = r \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

If G be the moment of the resistance, l the whole length of the cylinder, we have, putting $n = \frac{\pi}{\tau}$,

$$G = \frac{\pi^2 k' \rho a^2 l^3}{12\tau} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}\,;$$

whence

$$\ln\left[\frac{1}{(1-m)}\right] = \frac{\pi^2 k' \rho a^2 l^3}{24 I}\,, \tag{3.19}$$

I being the moment of inertia.
Expressing I in terms of the same quantities as in the case of the disk, we get from Equation 3.1 and Equation 3.19

$$\log\left[\frac{1}{(1-m)}\right] = \log(e) \cdot \frac{\pi\mu' T l^3}{3R^2} \cdot \frac{g\rho}{W} \cdot \frac{T}{\tau} \cdot \mathfrak{m}^2\, k'\,, \tag{3.20}$$

and $g\rho$ is the weight of a cubic millimetre of water, or the 1000th part of a gramme. The numerical values of μ', T, R, W have been already given, but μ' must be reduced from square inches to square millimetres. The cylinders, of which three were tried in succession, had all the same length, namely, 249 millimetres. Their circumferences, calculated from their weights and expressed in millimetres, were 21.1, 11.2, and 0.87, and the time of four oscillations was 92^3, 91^3, 91^3. The values of $\mathfrak{m}$ calculated from these data by means of Equation 3.1 are 0.4332, 0.2312, and 0.01796. For the first and second of these values, $\mathfrak{m}^2\, k'$ may be obtained by interpolation from Table 2.1 given in chapter 2; for the third it will be sufficient to employ the second of Equation 2.115.

The following are the results:

	Cylinder, No. 1	Cylinder, No. 2	Cylinder, No. 3
m, by experiment	0.0400	0.0260	0.0136
m, by theory	0.0413	0.0291	0.0113
Difference	−0.0013	−0.0031	+0.0023

The differences between the results of theory and experiment are perhaps as small as could reasonably be expected, when it is considered that, notwithstanding the delicate nature of the experiments, the numerical values of two constants, m and p, had to be deduced from their results.

68. This memoir of COULOMB's contains also a notice of a set of experiments with disks and cylinders in which the water was replaced by oil. The experiments with disks shewed that with a given disk the arc of oscillation decreased in geometric progression, and that with different disks the moments of the resistances were as the fourth powers of the diameters. The absolute resistances were greater than in the case of water in the ratio of about 17.5 to 1. The details of COULOMB's experiments on cylinders oscillating in oil are entirely omitted. It is merely stated that on making the same cylinders as before, or shorter cylinders when the resistance was too great, oscillate in oil, it was found, conformably with the results obtained with planes, that the coherence of oil was to that of water as 17 to 1. The coherence is here supposed to be measured by that part of the resistance which is proportional to the first power of the velocity. On making a rough calculation of the ratio of the resistances to cylinders oscillating in oil and in water, on the supposition that $\sqrt{\mu'}$ for oil is to $\sqrt{\mu'}$ for water as 17.5 to 1, as would follow from the experiments on disks if the difference of the specific gravities of the two fluids be neglected, I found that the ratio in question ought to have been somewhere about 100 to 1, instead of only 17 to 1. It would seem from this that the theory of the present paper is not applicable to oil; but fresh experiments would be required before this point can be considered as established, on account of the theoretical doubt respecting the application of the formulæ of section 2.3, to extremely fine cylinders, especially in cases in which $\sqrt{\mu'}$ is large, so that $\mathfrak{m}$ is very small. It would be interesting to make out whether what I have called internal friction is or is not of the same nature as viscosity. COULOMB and DUBUAT apply the term *viscosity* to that property of water by virtue of which certain effects are produced which have been shewn in this paper to be perfectly explicable on the theory of internal friction; whereas POISSON, in one of his memoirs, expressly asserts that the terms in the equations of motion which result from what has been called in this paper internal friction belong to perfect fluids, and have nothing to do with viscosity[10]. POIS-

[10] *Journal de l'Ecole Polytechnique*, Tom. XIII. p. 95., 1831, https://gallica.bnf.fr/ark:

SON does not give the slightest hint as to the grounds on which he rested his opinion.

69. I come now to the experiments of DUBUAT, which are contained in an excellent work of his entitled *Principes d'Hydraulique*, of which the second edition was published in 1786. The first edition does not contain the experiments in question. DUBUAT justly remarked that the time of oscillation of a pendulum oscillating in a fluid is greater than it would be in vacuum, not only on account of the buoyancy of the fluid, which diminishes the moving force, but also on account of the mass of fluid which must be regarded as accompanying the pendulum in its motion; and even determined experimentally the mass of fluid which must be regarded as carried by the oscillating body in the case of spheres and of several other solids. Thus DUBUAT anticipated by about forty years the discovery of BESSEL; but it was not until after the appearance of BESSEL's memoir that DUBUAT's labours relating to the same subject attracted attention.
DUBUAT's method was as follows. Imagine a body suspended by a fine thread or wire and swung in vacuum, and let a be the length of the pendulum, reckoned from the centre of suspension to the centre of oscillation. Now imagine the same body swung in a fluid, in which its apparent weight is p, so that if P denote the weight of fluid displaced, the true weight of the body will be $p + P$. Since the moving force is diminished in the ratio of $p + P$ to p, if the inertia of the body were all that had to be overcome, it would be necessary to diminish the length of the pendulum in the same ratio, in order to preserve the same time of oscillation. But since the mass in motion consists not only of the mass of the body itself, but also of that of the fluid which it carries with it, the pendulum must be shortened still more, in order that the time of oscillation may be unaltered. Let l be the length of the pendulum so shortened, and $\mathfrak{n}$ (which for the same reason as before I write instead of DUBUAT's n) a factor greater than unity, such that $p + \mathfrak{n}\,P$ is the weight of the mass in motion; then

$$l = \frac{ap}{(p + \mathfrak{n}\,P)}\,, \quad \text{whence} \quad \mathfrak{n} = \frac{p}{P} \cdot \left(\frac{a}{l} - 1\right)\,. \tag{3.21}$$

DUBUAT's experiments on this subject consist of 44 experiments on spheres oscillating in water, (Tom. II. p. 236); 31 experiments on other solids oscillating in water, (p. 246); and 3 experiments on spheres oscillating in air, (p. 283). The following table contains a comparison of Equation 3.2 with DUBUAT's results for spheres oscillating in water. The value of $\sqrt{\mu'}$ employed in the calculation is 0.0564 inch English, or 0.05291 inch French.

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	τ	$\mathfrak{n}$ calc.	obs.	diff.
Sphere of lead. Diameter 1.0113 inches. Weight in water 2102 grains.	$\frac{1}{2}$	1.633	1.502	-0.131
	1	1.687	1.502	-0.185
	2	1.766	1.522	-0.244
	3	1.825	1.620	-0.205
Sphere of glass. 2 Diameter 2.645 inches. Weight in water 574 grains.	2	1.602	1.518	-0.084
	4	1.644	1.569	-0.075
	6	1.676	1.598	-0.078
Same sphere weighing in water 2102 grains.	1	1.572	1.515	-0.057
	2	1.602	1.516	-0.086
	3	1.624	1.523	-0.101
	4	1.644	1.546	-0.098
Same sphere weighing in water 4204 grains.	1	1.572	1.537	-0.035
	2	1.602	1.523	-0.079
	3	1.624	1.524	-0.100
	4	1.644	1.538	-0.106
Same sphere weighing in water 9216 grains.	$\frac{1}{2}$	1.551	1.449	-0.102
	1	1.572	1.372	-0.200
	2	1.602	1.494	-0.108
	3	1.624	1.494	-0.130
Sphere of wood. Diameter 4.076 inches. Weight in water 2102 grains.	2	1.566	1.507	-0.059
	3	1.581	1.547	-0.034
	4	1.593	1.547	-0.046
	6	1.614	1.567	-0.057
Same sphere weighing in water 4204 grains.	1	1.547	1.375	-0.172
	2	1.566	1.456	-0.110
	3	1.581	1.525	-0.056
	4	1.593	1.557	-0.036
	6	1.614	1.549	-0.065
Same sphere weighing in water 9216 grains.	1	1.547	1.57	+0.023
	2	1.566	1.553	-0.013
	3	1.581	1.59	+0.009
	4	1.593	1.583	-0.010
Another sphere of wood. Diameter $6\frac{2}{3}$ inches. Weight in water 2102 grains.	3	1.549	1.27	-0.279
	4	1.557	1.394	-0.163
	6	1.570	1.487	-0.083
	9	1.585	1.566	-0.019
	12	1.599	1.569	-0.030
	18	1.621	1.565	-0.056
Same sphere weighing in water 3204 grains.	10.85	1.594	1.634	+0.040
Same sphere weighing in water 4204 grains.	3	1.549	1.651	+0.102
	4	1.557	1.627	+0.070
	6	1.570	1.654	+0.084
	9	1.585	1.664	+0.079
	12	1.599	1.674	+0.075

Table 3.9.: DUBUAT's experiments on spheres oscillating in water

70. If we strike out the experiments with the large sphere, which cannot well be compared with theory for a reason which will be explained further on, it will be observed that in seven out of the eight groups of experiments left, the signs in the last column are regularly *minus*. The preponderance of negative errors could be destroyed by using a much smaller value of $\sqrt{\mu'}$ in the reduction. We have seen, however, that the value of $\sqrt{\mu'}$ deduced from COULOMB's experiments on the decrement of the arc of oscillation of disks satisfied almost exactly BESSEL's observations of the time of oscillation of a sphere about two inches in diameter oscillating in water. The very small errors which remained in this case had both the sign +, whereas in DUBUAT's experiments on the 1-inch and $2\frac{1}{2}$-inch spheres, the errors, which are far larger, have all the sign −. Since the experiments of DUBUAT and BESSEL, though made under similar circumstances, do not lead to the same result, it is of course impossible for any theory to satisfy them both. The numbers in the last column of the preceding table are, however, far too regular to be attributable to mere fortuitous errors of observation. If we suppose BESSEL's results to have been nearly exact, there must have been something in the mode either of making or of reducing DUBUAT's experiments which caused a tendency to error in one direction.

With respect to the reduction of the experiments it may be observed that the length l was measured from the centre of oscillation, whereas in Equation 3.2 it is supposed that the mass of which the weight is $k \cdot P$ or $(\mathfrak{n}-1) \cdot P$ is collected at the centre of the sphere. If h be the distance of the centre of the sphere from the axis of suspension, the observed value of $(\mathfrak{n}-1)$ ought in strictness to be increased in the ratio of h^2 to l^2, or the calculated value diminished in the ratio of l^2 to h^2, before comparing the results of theory and experiment. In the case of the loaded spheres especially, the theoretical value of $\mathfrak{n}$ would thus be a little diminished; but except in a very few cases, in which either l or $(a-l)$ is small, the diminution is hardly worth considering. After having been for a good while at a loss to account for the regular occurrence of rather large negative errors, the following occurred to me as the probable solution of the difficulty.

When a pendulum oscillates in water, the arc of oscillation rapidly decreases; this rapid diminution forms in fact the grand difficulty in experiments of this kind. In DUBUAT's experiments, it will be remembered, the suspending thread was lengthened or shortened till the time of oscillation was an exact number of seconds, or occasionally half a second. Now, it is probable that the observer occasionally gave the suspending thread a slight push as the pendulum was commencing its return, in order to keep the oscillations going for a sufficient time to allow of tolerable precision in rendering the time of oscillation equal to what it ought to be. If so, these pushes would slightly accelerate the oscillations, and therefore cause the length of thread fixed on by observation to be a little too great, which would make the effect of the water in retarding the oscillations appear a little too small. On inspecting the table of differences, it may be observed that sometimes when the same sphere differently loaded is swung in the same time as before, the numbers

in the table of differences are altered more than appears to be attributable to merely fortuitous errors of observation. This accords very well with the conjecture just mentioned, and seems difficult to account for in any other way, inasmuch as everything relating to the fluid must have been almost exactly the same in the two cases.
The occurrences of positive differences in the case of the large wooden sphere may be accounted for by the limitation of the fluid mass by the sides and bottom of the vessel, and by the free surface, which, except in the case of very short oscillations, would have much the same effect as a rigid plane, inasmuch as it would be preserved almost exactly horizontal by the action of gravity. The vessel which contained the water was 51 inches long and 17 broad, the water was 14 inches deep, and the spheres were plunged to about 3 inches below the surface, so that the effect of the confinement of the fluid mass would have been quite sensible in the case of such large spheres. If it be objected that the same sphere gave negative differences in the case of the first group of experiments, it must be observed, that when the apparent weight of so large a sphere was only 2102 French grains, the resistance would quickly have caused the oscillations to subside if an extraneous force had not frequently been applied.

71. In DUBUAT's experiments on spheres oscillating in air, the lightness of the fluid was compensated by the extreme lightness of the spheres, which were composed, the first two of paper, and the third of gold-beater's skin. In the following table the diameter $2a$ of the sphere is expressed in French inches. The value of $\sqrt{\mu'}$ employed in the reduction is the same as was before used in the reduction of observations made in air, namely 0.116 inch English, or 0.1088 inch French.

No.	$2a$	τ	$\mathfrak{n}$ calc.	$\mathfrak{n}$ obs.	Diff.
337	4.0416	1.51	1.61	1.51	−0.10
338	6.625	1.84	1.57	1.63	+0.06
339	17.25	3.625	1.53	1.54	+0.01

Table 3.10.: DUBUAT's experiments on light spheres oscillating in air

The differences certainly appear very small when the delicacy of the experiments and the simplicity of the apparatus employed are considered.

72. The only comparison yet made in this section between theory and observation in the case of pendulum experiments, consists in comparing the observed times of vibration with the results calculated with an assumed value of $\sqrt{\mu'}$. But according to theory we ought to be able, without assigning a particular value to any new disposable constant, to calculate the rate of decrease of the arc of vibration. I

have not met with any experiments made with a view of investigating the decrease in the arc of vibration in the case of extremely small vibrations, such as those employed in pendulum experiments. The experiments of NEWTON and others, in which the arc of vibration was so large that the resistance depended mainly on the square of the velocity, would be quite useless for my purpose. The pendulum experiments of BESSEL and BAILY contain however the requisite information, or at least some portion of it, for the arcs are registered for the sake of giving the data for calculating the small reduction to indefinitely small vibrations.
In BESSEL's experiments the arc is registered for the end of equal intervals of time during the motion. The number of such registrations in one experiment amounts in some cases to eleven, and is never less than three. So far the observations are just what are wanted; but there are other causes which prevent an exact comparison between theory and experiment. In the first place the spheres were swung so close to the back of the frame that the increase of resistance due to the confinement of the air must have been very sensible. In the second place the effect of the wire must have been very sensible, especially in the case of the long pendulum. For Table 2.1 shews that for the wire (for which $\mathfrak{m}$ is very small) the value of k' is much larger than that of k, whereas for spheres of the size of those employed, when the time of oscillation is only one or two seconds, k' is a good deal smaller than k. Hence, if the formulæ of that section are applied to such fine wires, the effect of the wire on the arc of vibration would be much greater than its effect on the time of vibration, and therefore would be quite sensible. But it has been shewn in section 2.4, that the effect of the wire in diminishing the arc of vibration is probably greater than would be given by the formula, and therefore the uncertainty depending on the wire is likely to amount to a very sensible fraction of the whole amount. Again, since BESSEL's experiments were all made in air, no data are afforded whereby to eliminate the portion of the observed result which was due to friction at the point of support, imperfect elasticity of the wire, or gradual dissipation of vis viva by communication of motion to the supporting frame. Moreover in the case of the long pendulum the observations were made with rather too large arcs, for the law of the decrease of the arc of vibration deviated sensibly from that of a geometric progression. In BAILY's experiments, only the initial and final arcs are registered, and not even those in the case of the "additional experiments". Hence these experiments do not enable us to make out whether it would be sufficiently exact to suppose the decrease to take place in geometric progression. Moreover, the final arc was generally so small, that a small error committed in the measurement of it would cause a very sensible error in the rate of decrease concluded from the experiment. For these reasons it would be unreasonable to expect a near accordance between the formulæ and the results of the experiments of BESSEL and BAILY. Still, the formulæ might be expected to give a result in defect, and yet not so much in defect as not to form a large portion of the result given by observation. On this account it will not be altogether useless to compare theory and observation with reference to the decrement of the arc of vibration.

73. Let us first consider the case of a sphere suspended by a fine wire. Let the notation be the same as was used in investigating the expression for the effect of the air on the time of vibration, except that the factors k', k'_1 come in place of k, k_1. Considering only that part of the resistance which affects the arc of vibration, we have for the portions due respectively to the sphere and to the element of the wire whose length is $\mathrm{d}s$, and distance from the axis of suspension s,

$$k'M'n \cdot (l+a) \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}\,,$$
$$k'_1 \cdot \frac{M'_1}{l}\,\mathrm{d}s \cdot ns \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}\,,$$

and if we take the moment of the resistance, and divide by twice the moment of inertia, the coefficient of $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ in the result, taken negatively, and multiplied by t, will be the index of e in the expression for the arc. Hence if a_0 be the initial arc of vibration, and a_t the arc at the end of the time t

$$\ln(a_0) - \ln(a_t) = \frac{k'M' \cdot (l+a)^2 + \frac{1}{3}k'_1M'_1l^2}{M \cdot (l+a)^2 + \frac{1}{3}M_1l^2} \cdot \frac{\pi t}{2\tau}\,, \tag{3.22}$$

$M' \cdot (l+a)^2$ being as before taken for the moment of inertia of the sphere, which will be abundantly accurate enough. If then we put $\mathfrak{l}$ for the Napierian logarithm[11] of the ratio of the arc at the beginning to the arc at the end of an oscillation, we must put $t = \tau$ in Equation 3.22, whence, neglecting the effect of the wire, we obtain

$$\mathfrak{l} = \frac{\pi k'}{2} \cdot \frac{\sigma}{S}\,. \tag{3.23}$$

If now $\Delta k'$ be the correction to be applied to k' in this formula on account of the wire, since k', k'_1 are combined together in the expression for the arc just as k, k_1 in the expression for the time, we get

$$\Delta k' = \frac{k'_1}{k_1} \cdot \Delta k\,, \tag{3.24}$$

and the approximate Equation 2.115 give

$$\Delta k' = -\frac{4L}{\pi} \cdot \Delta k\,, \tag{3.25}$$

whence the numerical value of $\Delta k'$ is easily deduced from that of Δk, which has

[11] See Appendix F at the end.

been already calculated. We get also from Equation 2.52

$$k' = k - \frac{1}{2} + \frac{4}{9} \cdot \left(k - \frac{1}{2}\right)^2 , \tag{3.26}$$

whence k' may be readily deduced from k, which has been already calculated.

74. Before comparing these formulæ with BESSEL's experiments, it will be proper to enquire how far the latter are satisfied by supposing the arcs of oscillation to decrease in geometric progression. In BESSEL's tables the arc is registered in the column headed μ. This letter denotes the number of French lines read off on a scale placed behind the wire, and a little above the sphere, and is reckoned from the position of instantaneous rest of the wire on one side of the vertical to the corresponding position on the other side. The distance of the scale from the axis of suspension being given, as well as the correction to be applied to μ on account of parallax, the arc of oscillation may be readily deduced. However, for our present purpose, any quantity to which the arc is proportional will do as well as the arc itself, and μ, though strictly proportional to the tangent of the arc, may be regarded as proportional to the arc itself, inasmuch as the initial arc usually amounted to only about $50'$ on each side of the vertical.
Now we may form a very good judgment as to the degree of accuracy of the geometric formula by comparing the arc observed in the middle of an experiment with the geometric mean of the initial and final arcs. I have treated in this way BESSEL's experiments, Nos. 1, 2, 3, 4, and 5. Each of these is in fact a group of six experiments, four with the long pendulum and two with the short, so that the whole consists of 20 experiments with the long pendulum, and 10 with the short. In the case of the long pendulum, the observed value of μ regularly fell short of the calculated value, and that by a tolerably constant quantity. The mean difference amounted to 0.688 line, and the mean error in this quantity to 0.109. This mean error was not due entirely to errors of observation, or variations in the state of the air, &c., but partly also to slight variations in the initial arc, larger differences usually accompanying larger initial arcs. The initial arc usually corresponded to $\mu = 39$ or 40 lines, and the final to $\mu = 15$ or 16 lines. In the case of the short pendulum, the differences in 8 cases out of 10 had the same sign as before. The mean difference was 0.025, and the mean error 0.043. The arcs of oscillation were nearly the same as before; but inasmuch as the axis of suspension was nearer to the scale than before, the initial value of μ was only about 12 or 13 lines, and the final value about 7 lines. When the results of some of the experiments were laid down on paper, by abscissas taken proportional to the times and ordinates to the logarithms of μ, it was found that in the case of the long pendulum the line so drawn was decidedly curved, the concavity being turned toward the side of the positive ordinates. The curvature of the line belonging to the short pendulum could hardly be made out, or at least separated from the effects of errors of observation.

The experiments 9, 10, 11, having been treated numerically in the same way as the experiments 1-5, led to much the same result. In the 16 experiments with the ivory sphere and short pendulum contained in the experiments Nos. 12, 13, 14, and 15, the excess of the calculated over the observed value of μ was more apparent, the mean excess amounting to 0.129. The reason of this probably was, that the observations with the ivory sphere were made through a somewhat wider range of arc than those with the brass sphere.
It appears then that at least in the case of the long pendulum a correction is necessary, in order to clear the observed decrease in the arc of oscillation from the effect of that part of the resistance which increases with the arc more rapidly than if it varied as the first power of the velocity, and so to reduce the observed rate of decrease to what would have been observed in the case of indefinitely small oscillations.

75. In COULOMB's experiments it appeared that the resistance was composed of two terms, one involving the first power, and the other the square of the velocity. If we suppose the same law to hold good in the present case, and denote the amplitude of oscillation at the end of the time t, measured as an angle, by α, we shall obtain

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -A\alpha - B\alpha^2\,, \tag{3.27}$$

where A and B are certain constants. We must now endeavour to obtain A from the results of observation. Since the substitution for α of a quantity proportional to α will only change the constant B in Equation 3.27, and the numerical value of this constant is not required for comparison with theory, we may substitute for α the number of lines read off on the scale as entered in BESSEL's tables in the columns headed μ.
I have employed four different methods to obtain A from the observed results. The one I am about to give is the shortest of the four, and is sufficiently accurate for the purpose.
The Equation 3.27 gives after dividing by α

$$\frac{\mathrm{d}\log(\alpha)}{\mathrm{d}t} = -A - B\alpha\,. \tag{3.28}$$

Now, as has been already observed, the arcs of vibration decrease nearly in geometric progression. If this law were strictly true, we should have

$$\alpha = \alpha_0 \cdot \left(\frac{\alpha_2}{\alpha_0}\right)^{\frac{t}{T}} \tag{3.29}$$

where α_0 denotes the initial and α_2 the final arc, and T denotes the whole time of observation. We may, without committing any material error, substitute this value of α in the last term of Equation 3.28. The magnitude of the error we thus

commit is not to be judged of merely by the smallness of B. The approximate Equation 3.29 is rather to be regarded as a well-chosen formula of interpolation, and in fact $\frac{1}{T} \cdot \ln\left(\frac{\alpha_0}{\alpha_2}\right)$ differs very sensibly from A. Making now this substitution in Equation 3.28, integrating, and after integration restoring α in the last term by means of Equation 3.29, we get

$$\log(\alpha) = -At - \frac{BT\alpha}{\log(\alpha_2) - \log(\alpha_0)} + C\,, \tag{3.30}$$

C being an arbitrary constant. To determine the three constants A, B, C, let α_1 be the arc observed at the middle of the experiment, apply the last equation to the arcs α_0, α_1, α_2, and take the first and second differences of each member of the equation. Let Δ_1 denote the sum of the two first differences, so that $\Delta_1 t$ is the same thing as T. Then we may take for the two equations to determine A and B

$$\Delta_1 \cdot \log(\alpha_0) = -A\Delta_1 t - \frac{B\Delta_1 t \cdot \Delta_1 \alpha_0}{\Delta_1 \cdot \log(\alpha_0)}\,,$$
$$\Delta^2 \cdot \log(\alpha_0) = -\frac{B\Delta_1 t \cdot \Delta^2 \alpha_0}{\Delta^2 \cdot \log(\alpha_0)}\,.$$

Eliminating B, and passing from Napierian to common logarithms, which will be denoted by log, we get

$$A = -\frac{\Delta_1 \cdot \log(\alpha_0)}{\log(e) \cdot \Delta_1 t} \cdot \left\{1 - \frac{\Delta^2 \cdot \log(\alpha_0) \cdot \Delta_1 \alpha_0}{\Delta_1 \cdot \log(\alpha_0) \cdot \Delta^2 \alpha_0}\right\}\,. \tag{3.31}$$

If we suppose the part of $-\frac{\mathrm{d}\alpha}{\mathrm{d}t}$ which does not vary as the first power of α to be $\alpha^2\phi'(\alpha)$ instead of $B\alpha^2$, we shall get in the same way

$$A = \frac{-\Delta_1 \cdot \log(\alpha_0)}{\log(e) \cdot \Delta_1 t} \cdot \left\{1 - \frac{\Delta^2 \cdot \log(\alpha_0) \cdot \Delta_1 \phi(\alpha_0)}{\Delta_1 \cdot \log(\alpha_0) \cdot \Delta^2 \phi(\alpha_0)}\right\}\,. \tag{3.32}$$

76. I have not attempted to deduce evidence for or against the truth of Equation 3.27 from Bessel's experiments. The approximate Equation 3.29 so nearly satisfied the observations, that almost any reasonable formula of interpolation which introduced one new disposable constant would represent the experiments within the limits of errors of observation. It may be observed, that the factor outside the brackets in Equation 3.31 and Equation 3.32 is the first approximate value of A got by using only the initial and final arcs, and supposing the arcs to decrease in geometric progression. In the case of the long pendulum, the value of A, corrected in accordance with Equation 3.32, would be very sensibly different according as we supposed $\phi(\alpha)$ to be equal to $B\alpha$, in which case Equation 3.32 would reduce itself to Equation 3.31, or equal to $B\alpha^2$. In the case of the long pendulum with the brass sphere, the corrected value of A, deduced from Equation 3.31, was equal

to about 0.77 of the first approximate value.
I have not considered it necessary to go through all BESSEL's experiments, as it was not to be expected that the formula should account for the whole observed decrement. I have only taken four experiments for each kind of pendulum, namely, I. a, b, e, and f for the long pendulum with the brass sphere; I. c and d and II. c and d for the short pendulum with the brass sphere; XII. a, b, c, and d for the long pendulum with the ivory sphere, and XII. a′, b′, c′, and d′ for the short pendulum with the ivory sphere. Equation 3.31 gave the following results. First case, $\log(e) \cdot \tau A = 0.0000759$; mean error = 0.0000020. Second case, $\log(e) \cdot \tau A = 0.0000504$; mean error = 0.0000075. Third case, $\log(e) \cdot A = 0.000631$; mean error = 0.000046. Fourth case, $\log(e) \cdot A = 0.000167$; mean error = 0.000074. Now $\mathfrak{l} = \tau A$ and therefore, to get the values of $\mathfrak{l}$ deduced from experiment, it will be sufficient to divide the numbers above given by the modulus of the common system of logarithms. The theoretical value of $\mathfrak{l}$ will be got from Equation 3.23, if we add to k' the correction $\Delta k'$ depending upon the wire. The following are the results:

		long p. brass s.	short p. brass s.	long p. ivory s.	short p. ivory s.
1000000 $\mathfrak{l}$ for sphere alone in an unlimited mass of fluid, by theory	...	67	50	298	222
additional for wire	...	27	9	114	39
		94	59	412	261
1000000 $\mathfrak{l}$ by experiment	...	175	116	1453	384

It appears then that the calculated rate of decrease of the arc amounts on the average to about half the rate deduced from observation. This is about what we might have expected, considering the various circumstances, all tending materially to augment the rate of decrease, which were not taken into account in the calculation.

77. Of BAILY's pendulums I have compared the following with theory in regard to the decrement of the arc of vibration. No. 1 (the $1\frac{1}{2}$-inch platina sphere), experiments 1 to 8; No. 3 (the brass $1\frac{1}{2}$-inch sphere), experiments 9 to 16; No. 6 (the 2-inch brass sphere), experiments 33 to 40; No. 21 (the 0.410-inch long copper cylindrical rod), experiments 109 to 112; and No. 35-38 (the $1\frac{1}{2}$-inch long brass tube), experiments 167 to 174. I have not thought it worth while to compute the results obtained with the other $1\frac{1}{2}$-inch and 2-inch spheres, inasmuch as they were of the same size as the brass spheres, and moreover the observation of the decrement of the arc was not the object BAILY had in view in making the experiments. The 3-inch sphere, and all the other cylindrical rods and combinations of cylindrical rods and spheres, belong to the "additional experiments" for which the arcs are

not given.

The mode of performing the calculation will best be explained by an example. Take, for instance, the pair of experiments Nos. 1 and 2. In No. 1 the total interval was 4.22 hours, the initial arc was 0°.77, the final arc 0°.29, the mean height of the barometer 30.24 inches, and the temperature about $38\frac{1}{2}^{\circ}$ F. The difference of the common logarithms of the initial and final arcs is 0.424, and this divided by the total interval gives 0.1005 for the difference of logarithms for one hour. The second experiment, treated in a similar way, gives 0.0352, which expresses the effect of friction at the point of support, communication of motion to the support itself, &c., together with the resistance of highly rarefied air at a pressure of only 0.97 inch of mercury. Since we have reason to believe that μ' is independent of the density, we may get the effect of air at a pressure of 30.24 - 0.97 or 29.27 inches of mercury by subtracting 0.0352 from 0.1005, which gives 0.0653. Reducing to 29 inches of mercury for convenience of comparison, we get 0.0649. Each pair of experiments is to be treated in the same way. Since the temperature was nearly the same in the experiments made with the same pendulum, we may suppose it constant, and equal to the mean of the temperatures in the experiments made under the full atmospheric pressure. The experiments reduced consist of four pair for each pendulum, except No. 21, for which only two pair were performed. The following are the results. For the $1\frac{1}{2}$-inch platina sphere 0.0644, mean error 0.0044. For the $1\frac{1}{2}$-inch brass sphere 0.180, mean error 0.024. For the 2-inch brass sphere 0.094, mean error 0.013. For the copper rod 0.486, mean error 0.113. For the brass tube the results were 0.145, 0.363, 0.338, 0.305. Rejecting the first result as anomalous, and taking the mean of the others, we get 0.335, mean error 0.030. To obtain ɬ from the mean results above given we have only to divide by 3600 times the modulus, and multiply by τ, and for the experiments with spheres we may suppose $\tau = 1$.

The mode of calculating ɬ from theory in the case of a sphere suspended by a fine wire has already been explained. For the sake of exhibiting separately the effect of the wire, I will give one intermediate step in the calculation.

	1.44-inch sphere.	1.46-inch sphere.	2.06-inch sphere.
k', for sphere alone ...	0.326	0.320	0.220
$\Delta k'$, the correction for the wire ...	0.130	0.130	0.045
Total, to be substituted in Equation 3.23 ...	0.456	0.450	0.265

Equation 3.22, which applies to a sphere suspended by a wire, will be applicable to a long cylindrical rod if we suppose $M = 0$. Hence the same Equation 3.23 that has been used for a sphere may be applied to a cylindrical rod if we suppose k' to refer to the rod. For the copper rod $k' = 1.107$, and for the tube $k' = 0.2561$. The

following are the results for the three spheres and two cylinders.

	No. 1.	No. 3.	No. 6.	No. 21.	Nos. 35-38.
1000000 ł, from experiment ...	41	115	60	315	206
1000000 ł, from theory ...	39	106	60	237	156
Difference ...	+2	+9	0	+78	+50

It appears that the experiments with spheres are satisfied almost exactly. The differences between the results of theory and observation are much larger in the case of the long cylinders. Large as these differences appear, they are hardly beyond the limits of errors of observation, though they would probably be far beyond the limits of errors of observation in a set of experiments performed on purpose to investigate the decrement of the arc of vibration. It was to be expected beforehand that the results of calculation would fall short of those of observation, inasmuch as only two arcs were registered in each experiment, so that no data were afforded for eliminating the effect of that part of the resistance which did not vary as the first power of the velocity.

78. I have now finished the comparison between theory and experiment, but before concluding this section 3.1 I will make a few general remarks.
When a new theory is started, it is proper to enquire how far the theory does violence to the notions previously entertained on the subject. The present theory can hardly be called new, because the partial differential equations of motion were given nearly thirty years ago by NAVIER, and have since been obtained, on different principles, by other mathematicians; but the application of the theory to actual experiment, except in some doubtful cases relating to the discharge of liquids through capillary tubes, and the determination of the numerical value of the constant μ', are, I believe, altogether new. Let us then, in the first instance, examine the magnitude of the tangential pressure which we are obliged by theory to suppose capable of existing in air or water.
For the sake of clear ideas, conceive a mass of air or water to be moving in horizontal layers, in such a manner that each layer moves uniformly in a given horizontal direction, while the velocity increases, in going upwards, at the rate of one inch per second for each inch of ascent. Then the sliding in the direction of a horizontal plane is equal to unity, and therefore the tangential pressure referred to a unit of surface is equal to μ or $\mu'\rho$. The absolute magnitude of this unit sliding evidently depends only on the arbitrary unit of time, which is here supposed to be a second. In the case supposed, it will be easily seen that the particles situated at one instant in a vertical line are situated at the expiration of one second in a straight line inclined at an angle of 45° to the horizon. Equating the tangential pressure $\mu'\rho$ to the normal pressure due to a height h of the fluid, we get $h = \frac{\mu'}{g}$, g being the

force of gravity. Putting now $g = 386$, $\mu' = (0.116)^2$ for air, $\mu' = (0.0564)^2$ for water, we get $h = 0.00003486$ inch for air, and $h = 0.000008241$ inch for water, or about the one thirty-thousandth part of an inch for air, and less than the one hundred-thousandth part of an inch for water. If we enquire what must be the side of a square in order that the total tangential pressure on a horizontal surface equal to that square may amount to one grain, supposing the density of air to be to that of water as 1 to 836, and the weight of a cubic inch of water to be 252.6 grains, we get 25 feet 8 inches for air, and 1 foot 10 inches for water. It is plain that the effect of such small forces may well be insignificant in most cases.

79. In a former paper I investigated the effect of internal friction on the propagation of sound, taking the simple case of an indefinite succession of plane waves, see page 94. It appeared that the effect consisted partly in a gradual subsidence of the motion, and partly in a diminution of the velocity of propagation, both effects being greater for short waves than for long. The second effect, as I there remarked, would be contrary to the result of an experiment of M. Biot's, unless we supposed the term expressing this effect to be so small that it might be disregarded. I am now prepared to calculate the numerical value of the term in question, and so decide whether the theory is or is not at variance with the result of M. Biot's experiment.

According to the expression given in the paper just mentioned, we have for the proportionate diminution in the velocity of propagation

$$\frac{8\pi^2\mu'^2}{9\lambda^2V^2}\,,$$

λ being the length of a wave, and V the velocity of sound. To take a case as disadvantageous as possible, suppose λ only equal to one inch, which would correspond to a note too shrill to be audible to human ears. Taking the velocity of sound in air at 1000 feet per second, there results for the common logarithm of the expression above written $\overline{11}.0428$, so that a wave would have to travel near 100000000000 inches, or about 1578000 miles, before the retardation due to friction amounted to one foot. It is plain that the introduction of internal friction leaves the theory of sound just as it was, so far as the velocity of propagation is concerned, at least if the sound be propagated in free air.

The effect of friction on the intensity of sound depends on the first power of μ'. In the case of an indefinite succession of plane waves, it appears that during the time t the amplitude of vibration is diminished in the ratio of 1 to e^{-ct}, and therefore the intensity in the ratio of 1 to e^{-2ct}, where

$$c = \frac{8\pi^2\mu'}{3\lambda^2}\,.$$

Putting $\lambda = 1$ and $t = 1$ we get 1 to 0.4923, or 2 to 1 nearly, for the ratio in which

the intensity is altered during one second in the case of a series of waves an inch long. The rate of diminution decreases very rapidly as the length of wave increases, so that in the case of a series of waves one foot long the intensity is altered in one second in the ratio of 1 to 0.995095, or 201 to 200 nearly. It appears then that in all ordinary cases the diminution of intensity due to friction may be neglected in comparison with the diminution due to divergence. If we had any accurate mode of measuring the intensity of sound it might perhaps be just possible, in the case of shrill sounds, to detect the effect of internal friction in causing a more rapid diminution of intensity than would correspond to the increase of distance from the centre of divergence.

3.2. Suggestions with reference to future experiments

80. I am well aware that the mere proposal of experiments does not generally form a subject fit to be brought before the notice of a scientific society. Nevertheless, as it frequently happens in the division of labour that one person attends more to the theoretical, another to the experimental investigation of some branch of science, it is not always useless for the theorist to point out the nature of the information which it would be most important to obtain from experiment. I hope, therefore, that I may be permitted to offer a few hints with reference to experiments in which the theory of the internal friction of fluids is concerned. I shall omit all details, since they would properly come in connexion with the experiments.
Experiments with which the theory of internal friction in fluids has more or less to do may be performed for either of the following objects: first, to test still further the truth of the theory; secondly, to determine the index of friction of various gases, liquids, or solutions; to investigate the dependance of the index of friction of a gas on its state of pressure, temperature, and moisture; or to endeavour to make out the law according to which the index of friction of a mixture of gases depends upon the indices of friction of the separate gases; thirdly, to measure the length of the seconds' pendulum, or its variation from one part of the earth's surface to another.

81. *First object.* The theory has been already put to a pretty severe test by means of the experiments of Baily and others. Nevertheless there are some uncertainties in the comparison of theory and experiment arising from the influence of modifying causes of which the effect could only be estimated from theory, and yet was not so small as to be merged in errors of observation. Moreover, experiments on the decrement of the arc of vibration are almost wholly wanting. The following system of pendulums, meant to be swung in air and in vacuum, would afford a very good test of the theory.

- No. 1. A 2-inch or $1\frac{1}{2}$-inch sphere swung with a fine wire.
- No. 2. A very small sphere swung with the same kind of wire.
- No. 3. A long cylindrical rod, a few tenths of an inch in diameter.
- No. 4. A cylinder only three or four inches long, of the same diameter as No. 3, swung with the same kind of wire as No. 1.

The vacuum tube ought to be of sufficient size to render the estimated correction for confined space less than, or at most comparable with, errors of observation. The vacuum apparatus used by Col. Sabine would do very well. If the vacuum tube be not of sufficient size, it ought to admit of removal, and to be removed when the pendulums are swung in air.

In all the experiments the arc of oscillation ought to be carefully observed several times during the motion, the observation of the arc being quite as important for the purposes of theory as the observation of the time. Indeed, if it should be inconvenient to observe the time, the observation merely of the arc would be very valuable as a test of theory. In that case an approximate value of the time of oscillation in air would be required.

In the system proposed, Nos. 1 and 3 are the principal pendulums, Nos. 2 and 4 are introduced for the sake of making certain small corrections to the results of Nos. 1 and 3. No. 2 is meant for the elimination from No. 1 of the effect of the wire, and No. 4 for the elimination from No. 3 of the effect of the resistance experienced by a small portion of the rod near its end. The times of vibration of the four pendulums ought to be nearly the same, although for that purpose slightly different lengths of wire would be required in Nos. 1, 2, and 4.

It follows from theory that for a given pendulum the factor $\mathfrak{n}$ is a function of the time of vibration. This is a result which seems to have been hardly so much as suspected by those who were engaged in pendulum experiments, or at most to have been mentioned as a mere possibility[12], and therefore it might be thought advisable to verify it by direct experiment. For my own part I regard it as so

[12]It should be observed however that in a subsequent memoir (*Astronomische Nachrichten*, No. 223, p. 106), Bessel deduced from other experiments that the value of k was larger for the long than for the short pendulum.

intimately connected with the fundamental principles of the theory, that if the theory be confirmed in other respects I think this result may be accepted on the strength of theory alone. The direct comparison with experiment would be inconvenient, because it would require a clock which kept excellent time, and yet admitted of being adjusted so as to make widely different numbers of vibrations in a day. The result could, however, be confirmed indirectly by observing the arc of vibration, an observation which is as easy with one time of vibration as with another.

82. *Second object.* According to theory, the index of friction may be deduced from experiments either on the arc or on the time of vibration. It must be left to observation to decide which give the more consistent results. Should the results obtained from the arc appear as trustworthy as those obtained from the time, it would apparently be much the easiest way of determining μ' for an elastic fluid to observe the arc, because no particular accuracy would then be required in the observation of time. As to the form of the pendulum, a cylindrical rod would apparently be the best if only a single pendulum were employed. The observation of the arc seems the only practicable way of determining the influence of temperature on the index of friction, unless the pendulum be extremely light, or unless the observer be content with the limited range of temperature which may be procured by making observations at different times of year. For in an apparatus artificially heated or cooled, it would be difficult to prevent small unknown variations of temperature, which would cause variations in the rate of vibration, in consequence of the expansion and contraction of the pendulum; and these variations would vitiate the result of the experiment, so far as the time of vibration is concerned, because the effect of the gas on the time of vibration is deduced from the small difference between two large quantities which are directly observed. But the effect of the gas on the arc of vibration produces by far the greater part of the whole diminution observed, and therefore small fluctuations of temperature would not be of much consequence, except so far as they might occasion gentle currents; and even then it would not be very important, because the forces thence arising would not be periodic, and dependent upon the phase of vibration of the pendulum.
The grand difficulty which besets the observation of the time of vibration of a pendulum oscillating in a liquid consists in the rapidity with which the oscillations subside. The best form of a pendulum to oscillate in a liquid would be a sphere suspended by a fine wire. The vessel containing the liquid and the sphere immersed in it ought to be so large as to render the correction for confined space insensible. But the index of friction of a liquid would probably be better determined by experiments more of the nature of those of COULOMB, or perhaps by the slow discharge of liquids through narrow tubes.
Among the gases for which μ' ought to be determined experimentally should be mentioned coal-gas, on account of the practical application which it appears possible to make of the result to the laying down of gas-pipes. The calculation of the resistance in a circular pipe is very simple, and is given in Art. 9, page 98 of

my former paper. According to the equations of condition assumed in the present paper we must put $U = 0$, U denoting in that article the velocity close to the surface. It appears that the pressure spent in overcoming friction varies as the mean velocity divided by the square of the diameter of the pipe, or as the rate of supply divided by the fourth power of the diameter. This goes on the supposition that the motion is sufficiently slow to allow of our neglecting the pressure which may be spent in producing eddies, in comparison with that spent in overcoming what really constitutes internal friction.

83. *Third object.* With respect to experiments for determining the length of the seconds' pendulum, the theory of internal friction rather enables us to calculate for certain forms of pendulum the correction due to the inertia of the air than points out any particular mode of performing the experiments. Even the ordinary theory of hydrodynamics points out the importance of removing all obstacles to the free motion of the air in the neighbourhood of the pendulum if we would calculate from theory the whole correction for reduction to a vacuum.
Since the theoretical solution has been obtained in the case of a long cylindrical rod, or of such a rod combined with a sphere, we may regard a pendulum formed in this manner, and which is convertible in air, as also convertible in vacuum, for it is of small consequence whether the pendulum be or be not really convertible in vacuum, provided that if it be not we know the correction to be applied in consequence.

Part IV.

XLVII. On the Constitution of the Luminiferous Æther

The phænomenon of aberration may be reconciled with the undulatory theory of light, as I have already shown (*Philosophical Magazine*, vol. xxvii, p. 9), without making the violent supposition that the æther passes freely through the earth in its motion round the sun, but supposing, on the contrary, that the æther close to the surface of the earth is at rest relatively to the earth. This explanation requires us to suppose the motion of the æther to be such, that the expression usually denoted by $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential. It becomes an interesting question to inquire on what physical properties of the æther this sort of motion can be explained. Is it sufficient to consider the æther as an ordinary fluid, or must we have recourse to some property which does not exist in ordinary fluids, or, to speak more correctly, the existence of which has not been made manifest in such fluids, by any phænomenon hitherto observed? I have already attempted to offer an explanation on the latter supposition (*Philosophical Magazine*, vol xxix. p. 6). Professor Challis, in his last communication, has considered the æther as an ordinary fluid.

In my paper last referred to, I have expressed my belief that the motion for which $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, which would take place if the æther were like an ordinary fluid, would be unstable; I now propose to prove the same mathematically, though by an indirect method.

Even if we supposed light to arise from vibrations of the æther accompanied by condensations and rarefactions, analogous to the vibrations of the air in the case of sound, since such vibrations would be propagated with about 10,000 times the velocity of the earth, we might without sensible error neglect the condensation of the æther in the motion which we are considering. As far as the case in hand is concerned, Professor Challis might have regarded ρ as constant, and treated p as he has treated s. Suppose, then, a sphere to be moving uniformly in a homogeneous incompressible fluid, the motion being such that the square of the velocity may be neglected. There are many obvious phænomena which clearly point out the existence of a tangential force in fluids in motion, analogous in many respects to friction in the case of solids. When this force is taken into account, the equations of motion become (*Cambridge Philosophical Transactions*, vol. viii. p. 297)

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\rho \cdot \frac{\mathrm{d}u}{\mathrm{d}t} + \mu \cdot \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} \right) , \tag{1}$$

with similar equations for y and z. In these equations the square of the velocity is omitted, according to the supposition made above, ρ is considered constant, and the fluid is supposed not to be acted on by external forces. We have also the equation of continuity

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z} = 0\,, \tag{2}$$

and the conditions, (1) that the fluid at tile surface of the sphere shall be at rest relatively to the surface, (2) that the velocity shall vanish at an infinite distance.

For my present purpose it is not requisite that the equations such as Equation 1 should be known to be true experimentally; if they were even known to be false they would be sufficient, for they may be conceived to be true without mathematical absurdity. My argument is this. If the motion for which $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, which would be obtained from the common equations, were stable, the motion which would be obtained from Equation 1 would approach indefinitely, as μ vanished, to one for which $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ was an exact differential, and therefore, for anything proved to the contrary, the latter motion might be stable; but if, on the contrary, the motion obtained from Equation 1 should turn out totally different from one for which $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, the latter kind of motion must necessarily be unstable.

Conceive a velocity equal and opposite to that of the sphere impressed both on the sphere and on the fluid. It is easy to prove that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ will or will not be an exact differential after the velocity is impressed, according as it was or was not such before. The sphere is thus reduced to rest, and the problem becomes one of steady motion. The solution which I am about to give is extracted from some researches in which I am engaged, but which are not at present published. It would occupy far too much room in this Magazine to enter into the mode of obtaining the solution: but this is not necessary; for it will probably be allowed that there is but one solution of the equations in the case proposed, as indeed readily follows from physical considerations, so that it will be sufficient to give the result, which may be verified by differentiation.

Let the centre of the sphere be taken for origin; let the direction of the real motion of the sphere make with the axes angles whose cosines are l, m, n, and let ν be the real velocity of the sphere; so that when the problem is reduced to one of steady motion, the fluid at a distance from the sphere is moving in the opposite direction with a velocity ν. Let a be the sphere's radius: then we have to satisfy the general Equation 1 and Equation 2 with the particular conditions

$$u = 0\,, \quad v = 0\,, \quad w = 0\,, \quad \text{when} \;\; r = a\,; \tag{3}$$

$$u = -l \cdot \nu\,, \quad v = -m \cdot \nu\,, \quad w = -n \cdot \nu\,, \quad \text{when} \;\; r = \infty\,, \tag{4}$$

r being the distance of the point considered from the centre of the sphere. It will be found that all the equations are satisfied by the following values,

$$p = \Pi + \frac{3}{2} \cdot \mu\nu \cdot \frac{a}{r^3} \cdot (lx + my + nz)\ ,$$

$$u = \frac{3}{4}\nu \cdot \left(\frac{a}{r^3} - \frac{a^3}{r^5}\right) \cdot x \cdot (lx + my + nz) + l\nu \cdot \left(\frac{1}{4}\frac{a^3}{r^3} + \frac{3}{4}\frac{a}{r} - 1\right)\ ,$$

with symmetrical expressions for v and w. Π is here an arbitrary constant, which evidently expresses the value of p at an infinite distance. Now the motion defined by the above expressions does not tend, as μ vanishes, to become one for which

$u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, and therefore the motion which would be obtained by supposing $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ an exact differential, and applying to the æther the common equations of hydrodynamics, would be unstable. The proof supposes tHe motion in question to be steady; but as such it may be proved to be, if the velocity of the earth be regarded as uniform, and an equal and opposite velocity be conceived impressed both on the earth and on the æther. Hence the stars would appear to be displaced in a manner different from that expressed by the well-known law of aberration.

When, however, we take account of a tangential force in the æther, depending, not on relative velocities, or at least not on relative velocities only, but on relative displacements, it then becomes possible, as I have shown (*Philosophical Magazine*, vol. xxix. p. 6), to explain not only the perfect regularity of the motion, but also the circumstance that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, at least for the æther which occupies free space; for as regards the motion of the æther which penetrates the air, whether about the limits of the atmosphere or elsewhere, I do not think it prudent, in the present state of our knowledge, to enter into speculation; I prefer resting in the supposition that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential. According to this explanation, any nascent irregularity of motion, any nascent deviation from the motion for which $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, is carried off into space, with the velocity of light, by transversal vibrations, which as such are identical in their physical nature with light, but which do not necessarily produce the sensation of light, either because they are too feeble, as they probably would be, or because their lengths of wave, if the vibrations take place in regular series, fall beyond the limits of the visible spectrum, or because they are discontinuous, and the sensation of light may require the succession of a number of similar vibrations. It is certainly curious that the astronomical phænomenon of the aberration of light should afford an argument in support of the theory of transversal vibrations.

Undoubtedly it does violence to the ideas that we should have been likely to form *à priori* of the nature of the æther, to assert that it must be regarded as an elastic solid in treating of the vibrations of light. When, however, we consider the wonderful simplicity of the explanations of the phænomena of polarization when we adopt the theory of transversal vibrations, and the difficulty, which to me at least appears quite insurmountable, of explaining these phænomena by any vibrations due to the condensation and rarefaction of an elastic fluid such as air, it seems reasonable to suspend our judgement, and be content to learn from phænomena the existence of forces which we should not beforehand have expected. The explanations which I had in view are those which belong to the geometrical part of the theory; but the deduction, from dynamical calculations, of the laws which in the geometrical theory take the place of observed facts must not be overlooked, although here the evidence is of a much more complicated character.

The following illustration is advanced, not so much as explaining the real nature of the æther, as for the sake of offering a plausible mode of conceiving how the apparently opposite properties of solidity and fluidity which we must attribute to

the æther may be reconciled.

Suppose a small quantity of glue dissolved in a little water, so as to form a stiff jelly. This jelly forms in fact an elastic solid: it may be constrained, and it will resist constraint, and return to its original form when the constraining force is removed, by virtue of its elasticity; but if we constrain it too far it will break. Suppose now the quantity of water in which the glue is dissolved to be doubled, trebled, and so on, till at last we have a pint or a quart of glue water. The jelly will thus become thinner and thinner, and the amount of constraining force which it can bear without being dislocated will become less and less. At last it will become so far fluid as to mend itself again as soon as it is dislocated. Yet there seems hardly sufficient reason for supposing that at a certain stage of the dilution the tangential force whereby it resists constraint ceases all of a sudden. In order that the medium should not be dislocated, and therefore should have to be treated as an elastic solid, it is only necessary that the amount of constraint should be very small. The medium would however be what we should call a fluid, as regards the motion of solid bodies through it. The velocity of propagation of normal vibrations in our medium would be nearly the same as that of sound in water; the velocity of propagation of transversal vibrations, depending as it does on the tangential elasticity, would become very small. Conceive now a medium having similar properties, but incomparably rarer than air, and we have a medium such as we may conceive the æther to be, a fluid as regards the motion of the earth and planets through it, an elastic solid as regards the small vibrations which constitute light. Perhaps we should get nearer to the true nature of the æther by conceiving a medium bearing the same relation to air that thin jelly or glue water bears to pure water. The sluggish transversal vibrations of our thin jelly are, in the case of the æther, replaced by vibrations propagated with a velocity of nearly 200,000 miles in a second: we should expect, *à priori*, the velocity of propagation of normal vibrations to be incomparably greater. This is just the conclusion to which we are led quite independently, from dynamical principles of the greatest generality, combined with the observed phenomena of optics[1].

I take this opportunity of making a few remarks on my explanation of aberration (*Philosophical Magazine*, vol. xxvii, p. 9), more especially as Professor Challis's words at page 168 of the present volume would naturally lead to the idea, which, however, I believe was not intended, that I had only explained the change in the direction of the normal to a wave of light, so that something was wanting to complete, on the suppositions adopted, the explanation of aberration. To prevent misapprehension, I would observe, that in the explanation of aberration I here include the explanation of the rectilinear propagation of light, if the explanation of aberration be divided into two parts; the first, the explanation of rectilinear propagation; the second, the explanation of aberration on the assumption of rectilinear

[1]See the introduction to an admirable memoir by Green, On the Reflexion and Refraction of Light. –*Cambridge Philosophical Transactions*, vol. vii. p. 1.

propagation. To my own mind, the undulatory theory cannot be said to explain aberration unless it explains, either rectilinear propagation, or what is equivalent to it; for had the stars never been observed, I should have thought it excessively improbable that the path of a ray was rectilinear in the neighbourhood of the earth. As to the necessity for an explanation of aberration on any theory of light, I quite agree with Professor Challis, as he has stated. Indeed, if I ever appeared to differ from him on this point, it was not because I held a different opinion, but because I failed to catch his meaning.

In my first paper on aberration, it is true that I did not investigate the nature of the path of a ray of light in space; but this was only because the method I employed did not require any such investigation. I showed that, on the suppositions adopted, the path of a ray, not in space but relatively to the earth, was in the immediate neighbourhood of the observer directed to the apparent place of the heavenly body from which it came, that is, its place as affected by the observed aberration; and that was sufficient for my purpose. My explanation was not even deficient in consequence of not taking account of the light coming from the wire to which the star was referred; for according to my method, everything was reduced to the case in which the earth and the æther in its immediate neighbourhood are supposed to be at rest; so that there was no more occasion to notice *explicitly* the light coming from the wires, than there would have been if the earth had really been at rest. While, however, I would vindicate my explanation from any flaw or deficiency of reasoning (unless the not noticing formally and explicitly the light coming from the wires be regarded as such), I allow that, without investigation, I fancied the path of a ray in space to be curvilinear. It was first virtually proved by Professor Challis, though not explicitly stated, that the path was rectilinear throughout. Consequently the angle *peq* (*Philosophical Magazine*, vol. xxvii. p. 14), which I argued was insensible, is in fact zero. The method which consists in considering the rectilinear propagation of light as resulting from the supposition that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential, and then the law of aberration as resulting from the rectilinear propagation, instead of considering the whole at once, has the advantage of showing that we are at liberty to suppose the velocity of the æther at the surface of the earth to be of any amount relatively to the surface. I had not contemplated this case; for it was the precise object of my investigation to get rid of the apparent necessity of supposing the æther to be rushing through the air and through the earth itself as the earth moves round the sun.

Part V.

Appendix

A. Note A, Article 65.

Let us apply the general equations, Equation 2.2, and Equation 2.3 to the fluid surrounding a solid of revolution which turns about its axis, with either a uniform or a variable motion, supposing the fluid to have been initially either at rest, or moving in annuli about the axis of symmetry.

In the first place we may observe, that the fluid will always move in annuli about the axis of symmetry. For let P be any point of space, and L any line passing through P, and lying in a plane drawn through P and through the axis of symmetry; and at the end of the time t let u' be the velocity at P resolved along L. Now consider a second case of motion, differing from the first in having the angular velocity of the solid and the initial velocity of the fluid reversed, everything else being the same as before. It follows from symmetry, that at the end of the time t the velocity at P resolved along L will be equal to u', since the motion of the solid and the initial motion of the fluid, which form the data of the one problem, differ from the corresponding quantities in the other problem only as regards the distinction between one way round and the other way round, which has no relation to the distinction between to and fro in the direction of a line lying in a plane passing through the axis of rotation. But since all our equations are linear as regards the velocity, it follows that in the second problem the velocity will be the same as in the first, with a contrary sign, and therefore the velocity at P in the direction of the line L will be equal to $-u'$. Hence $u' = -u'$, and therefore $u' = 0$, and therefore the whole motion takes place in annuli about the axis of rotation.

Let the axis of rotation be taken for the axis of z; let ω be the angle which a plane passing through this axis and through the point P makes with the plane of xy, and let v' be the velocity at P. Then

$$\begin{aligned} u &= -v' \cdot \sin(\omega)\,, \\ v &= v' \cdot \cos(\omega)\,, \\ w &= 0\,, \end{aligned}$$

and all the unknown quantities of the problem are functions of t, z, and ϖ, where

$$\varpi = \sqrt{x^2 + y^2}\,.$$

Substituting in Equation 2.2 the above values of u, v, and w, and after differentiation putting $\omega = 0$, as we are at liberty to do, we get

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}\varpi} &= 0\,, \\ \frac{\mathrm{d}p}{\mathrm{d}z} &= 0\,, \\ \rho \cdot \frac{\mathrm{d}v'}{\mathrm{d}t} &= \mu \cdot \left(\frac{\mathrm{d}^2 v'}{\mathrm{d}z^2} + \frac{\mathrm{d}^2 v'}{\mathrm{d}\varpi^2} + \frac{1}{\varpi} \cdot \frac{\mathrm{d}v'}{\mathrm{d}\varpi} - \frac{v'}{\varpi^2} \right) \end{aligned} \tag{A.1}$$

The first two of these equations give $p =$ a constant, or rather $p =$ a function of t, which for the same reason as in Art. 7, page 141 we have a right to suppose it to be equal to zero. The third equation combined with the equations of condition serves to determine v'.

Now in the particular case of an oscillating disk, the Equation A.1 becomes according to the mode of approximation adopted in Art. 8, page 143,

$$\mu \cdot \frac{\mathrm{d}^2 v'}{\mathrm{d}z^2} = \rho \cdot \frac{\mathrm{d}v'}{\mathrm{d}t}\,, \tag{A.2}$$

which in fact is the same as the second of the Equation 2.8. The solution thus obtained is as we have seen

$$v' = \varpi \cdot f(z,t)\,, \tag{A.3}$$

f denoting a function the form of which there is no need to write down, which satisfies Equation A.2 when written for v'. Now it will be seen at once that Equation A.3 satisfies the exact Equation A.1, and therefore the approximate solution obtained by the method of Art. 8, page 143, is in fact exact, except so far as regards the termination of the disk at its edge, which is what it was required to prove.

Passing from semi-polar to polar co-ordinates, by putting

$$\begin{aligned} z &= r \cdot \cos(\theta)\,, \\ \varpi &= r \cdot \sin(\theta)\,, \end{aligned}$$

we get from Equation A.1, after writing $\mu'\rho$ for μ

$$\frac{\mathrm{d}^2 v'}{\mathrm{d}r^2} + \frac{2}{r} \cdot \frac{\mathrm{d}v'}{\mathrm{d}r} + \frac{1}{r^2 \cdot \sin(\theta)} \cdot \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sin(\theta) \cdot \frac{\mathrm{d}v'}{\mathrm{d}\theta} \right) - \frac{v'}{r^2 \cdot \sin^2(\theta)} = \frac{1}{\mu'} \cdot \frac{\mathrm{d}v'}{\mathrm{d}t}\,. \tag{A.4}$$

Suppose now the solid to be a sphere, having its centre at the origin. Let a be its radius, $\wp$ its angular velocity, and suppose the fluid initially at rest. Then v' is to

be determined from the general Equation A.4 and the equations of condition

$$\begin{aligned} v' &= 0 \quad \text{when} \quad t = 0\,, \\ v' &= a\wp \cdot \sin(\theta) \quad \text{when} \quad r = a\,, \\ v' &= 0 \quad \text{when} \quad r = \infty\,. \end{aligned}$$

All these equations are satisfied by supposing

$$v' = v'' \cdot \sin(\theta)\,,$$

v'' being a function of r and t only. We get from Equation A.4

$$\frac{\mathrm{d}^2 v''}{\mathrm{d}r^2} + \frac{2}{r} \cdot \frac{\mathrm{d}v''}{\mathrm{d}r} - \frac{2v''}{r^2} = \frac{1}{\mu'} \cdot \frac{\mathrm{d}v''}{\mathrm{d}t}\,. \tag{A.5}$$

If we suppose $\wp$ constant, v'' will tend indefinitely to become constant as t increases indefinitely, and in the limit $\frac{\mathrm{d}v''}{\mathrm{d}t} = 0$, whence we get from Equation A.5 and the equations of condition $v'' = a \cdot \wp$ when $r = a$, $v'' = 0$ when $r = \infty$,

$$\begin{aligned} v'' &= \frac{\wp \cdot a^3}{r^2}\,, \\ v' &= \frac{\wp \cdot a^3}{r^2} \cdot \sin(\theta)\,. \end{aligned}$$

This is the solution alluded to in Art. 8, page 96 of my paper *On the Theories of the Internal Friction of Fluids in motion, &c.*

B. Note B, Article 65.

Let us resume the problem of Art. 7, page 141, but instead of the motion of the plane being periodic, let us suppose that the plane and fluid are initially at rest, and that the plane is then moved with a constant velocity V, and let the notation be the same as in Art. 7.
The general Equation 2.8 remain the same as before, but the equations of condition become in this case

$$\begin{aligned} v &= 0 \quad \text{when} \quad t = 0 \quad \text{from} \quad x = 0 \quad \text{to} \quad x = \infty\,, \\ v &= V \quad \text{when} \quad x = 0 \quad \text{from} \quad t = 0 \quad \text{to} \quad t = \infty\,. \end{aligned}$$

By FOURIER's theorem and another theorem of the same kind, v may be expanded between the limits 0 and ∞ of x in the following form:

$$\begin{aligned} v = \frac{2}{\pi} \cdot \int_0^\infty \int_0^\infty \cos(\alpha x) \cos(\alpha x') \phi(x', t)\, \mathrm{d}x'\, \mathrm{d}\alpha \\ + \frac{2}{\pi} \cdot \int_0^\infty \int_0^\infty \sin(\alpha x) \sin(\alpha x') \psi(x', t)\, \mathrm{d}x'\, \mathrm{d}\alpha\,. \end{aligned} \tag{B.1}$$

In fact, v could be expanded by means of either of these expressions separately, and of course can be expanded in an infinite number of ways by the sum of the two. If however v had been expanded by means of the first expression alone, its derivatives with respect to x could not have been obtained by differentiating under the integral signs, inasmuch as the derivatives of an odd order do not vanish when at $x = 0$, but would have been given by certain formulæ which I have investigated in a former paper.[1] A similar remark applies to the second expansion, in consequence of the circumstance that v itself and its derivatives of an even order do not vanish with x. But by combining the two expansions we may obtain the derivatives of v, up to any order i that we please to fix on, by merely differentiating under the integral signs. For we may evidently express the finite function v, and that in an infinite number of ways, as the sum of two finite functions $\phi(x, t)$, $\psi(x, t)$, which like v vanish when $x = \infty$, and which are such that the odd derivatives of the first, and the even derivatives of the second, up to the order i, as well as $\psi(x, t)$ itself, vanish when $x = 0$. Substituting now in the second Equation 2.8 the expression for v

[1] On the critical values of the sums of periodic series. *Camb. Phil. Trans.* Vol. VIII. p. 533., see `https://books.google.de/books?id=_YnvAAAAMAAJ&hl=de&source=gbs_navlinks_s`, p. 237.

given by Equation B.1, we see that the equation is satisfied provided

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} + \mu'\alpha^2\phi = 0\,,$$
$$\frac{\mathrm{d}\psi}{\mathrm{d}t} + \mu'\alpha^2\psi = 0\,.$$

These equations give

$$\phi(x',t) = \chi(x') \cdot e^{-\mu'\alpha^2 t}\,,$$
$$\psi(x',t) = \sigma(x') \cdot e^{-\mu'\alpha^2 t}\,,$$

where χ and σ denote two new arbitrary functions. Substituting in Equation B.1, and then passing to the first of the equations of condition, we get

$$0 = \chi(x) + \sigma(x)\,,$$

whence $\sigma(x) = -\chi(x)$ and

$$\begin{aligned} v &= \frac{2}{\pi} \cdot \int_0^\infty \int_0^\infty \cos\left[\alpha \cdot (x + x')\right] \cdot e^{-\mu'\alpha^2 t} \chi(x')\,\mathrm{d}x'\,\mathrm{d}\alpha \\ &= \frac{1}{\sqrt{\pi\mu' t}} \cdot \int_0^\infty e^{-\frac{(x+x')^2}{4\mu' t}} \chi(x')\,\mathrm{d}x'\,. \end{aligned} \tag{B.2}$$

The second of the equations of condition requires that

$$\begin{aligned} V &= \frac{1}{\sqrt{\pi\mu' t}} \cdot \int_0^\infty e^{-\frac{x'^2}{4\mu' t}} \chi(x')\,\mathrm{d}x' \\ &= \frac{2}{\sqrt{\pi}} \cdot \int_0^\infty e^{-s^2} \chi\left(2s \cdot \sqrt{\mu' t}\right) \mathrm{d}s\,. \end{aligned}$$

Since the second member of this equation must be independent of t, we get $\chi(x') =$ a constant, and this constant must be equal to V, since

$$\frac{2}{\sqrt{\pi}} \cdot \int_0^\infty e^{-s^2}\,\mathrm{d}s = 1\,.$$

Substituting in Equation B.2 we get

$$v = \frac{V}{\sqrt{\pi\mu' t}} \cdot \int_0^\infty e^{-\frac{(x+x')^2}{4\mu' t}}\,\mathrm{d}x'\,. \tag{B.3}$$

For the object of the present investigation nothing is required but the value of $\frac{\mathrm{d}v}{\mathrm{d}x}$ for $x = 0$, which we may denote by $\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_0$. We get from Equation B.3

$$\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_0 = -\frac{V}{\sqrt{\pi\mu' t}}\,. \tag{B.4}$$

Now suppose the plane to be moved in any manner, so that its velocity at the end of the time t is equal to $f(t)$. We may evidently obtain the result for this case by writing $f'(t')\,\mathrm{d}t'$ for V, and $t - t'$ for t in Equation B.4, and integrating with respect to t'. We thus get

$$\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_0 = -\frac{1}{\sqrt{\pi\mu'}} \cdot \int_{-\infty}^{t} f'(t')\frac{\mathrm{d}t'}{\sqrt{t-t'}} = -\frac{1}{\sqrt{\pi\mu'}} \cdot \int_{0}^{\infty} f'(t-t_1)\frac{\mathrm{d}t_1}{\sqrt{t_1}}\,. \tag{B.5}$$

To apply this result to the case of an oscillating disk, let $r \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} = r \cdot F(t)$ be the velocity of any annulus, and G the moment of the whole force of the fluid on the disk. Then

$$G = 4\pi\mu'\rho \cdot \int_0^a r^2 \cdot \left(\frac{\mathrm{d}v_0}{\mathrm{d}x}\right)_0 \mathrm{d}r\,;$$

and $\left(\frac{\mathrm{d}v_0}{\mathrm{d}x}\right)_0$ will be got from Equation B.5 by substituting $r \cdot F(t)$ for $f(t)$. We find thus

$$G = -\sqrt{\pi\mu'} \cdot \rho a^4 \cdot \int_0^\infty F'(t-t_1)\frac{\mathrm{d}t_1}{\sqrt{t_1}}\,. \tag{B.6}$$

If we suppose the angular velocity of the disk to be expressed by $A \cdot \sin(nt)$, where A is constant, we must put $F(t) = A \cdot \sin(nt)$ in Equation B.6, and we should then get after integration the same expression for G as was obtained in Art. 8, page 143, by a much simpler process. Suppose, however, that previously to the epoch from which t is measured the disk was at rest, and that the subsequent angular velocity is expressed by $A_t \cdot \sin(nt)$, where A_t is a slowly varying function of t. Then

$$\begin{aligned} F(t) &= 0 \quad \text{when} \quad t < 0\,, \\ F(t) &= A_t \cdot \sin(nt) \quad \text{when} \quad t > 0\,. \end{aligned}$$

On substituting in Equation B.6 we get

$$G = -\sqrt{\pi\mu'} \cdot \rho a^4 n \cdot \int_0^t A_{t-t_1} \cdot \cos\left[n \cdot (t-t_1)\right] \frac{\mathrm{d}t_1}{\sqrt{t_1}}\,. \tag{B.7}$$

Now treating A_t as a slowly varying parameter, we get from a formula given by Mr Airy, and obtained by the method of the variation of parameters,

$$\frac{\mathrm{d}A_t}{\mathrm{d}t} = \frac{G}{I} \cdot \sin(nt)\,, \tag{B.8}$$

where I denotes the moment of inertia. In the expression for G we may replace A_{t-t_1} under the integral sign by A_t outside it, because A_t is supposed to vary so slowly that A_{t-t_1} does not much differ from A_t while t_1 is small enough to render the integral of importance. Making this simplification and substituting in Equation B.8 we get

$$\frac{1}{A_t} \cdot \frac{\mathrm{d}A_t}{\mathrm{d}t} = -c \cdot \sin(nt) \cdot \int_0^t \cos\left[n \cdot (t - t_1)\right] \frac{\mathrm{d}t_1}{\sqrt{t_1}}\,, \tag{B.9}$$

where

$$c = \sqrt{\pi\mu'} \cdot \rho a^4 n \cdot \frac{1}{I}\,.$$

If then A_0 be the initial and A the final value of A_t, we get from Equation B.9

$$\log\left(\frac{A_0}{A}\right) = c \cdot \int_0^t \left\{\sin(nt) \cdot \int_0^t \cos\left[n \cdot (t - t_1)\right] \frac{\mathrm{d}t_1}{\sqrt{t_1}}\right\} \mathrm{d}t\,. \tag{B.10}$$

Let now $A_0 + \Delta A_0$ be what A_0 would become if, while the final arc A and the whole time t remained the same, the motion had been going on for an indefinite time before the epoch from which t is measured, in which case the superior limit in the integral involved in the expression for G would have been ∞ in place of t. Then

$$\log\left(\frac{A_0 + \Delta A_0}{A}\right) = c \cdot \int_0^t \left\{\sin(nt) \cdot \int_0^\infty \cos\left[n \cdot (t - t_1)\right] \frac{\mathrm{d}t_1}{\sqrt{t_1}}\right\} \mathrm{d}t\,, \tag{B.11}$$

whence by subtracting, member from member, Equation B.10 from Equation B.11, we get

$$\log\left(\frac{A_0 + \Delta A_0}{A_0}\right) = c \cdot \int_0^t \left\{\sin(nt) \cdot \int_t^\infty \cos\left[n \cdot (t - t_1)\right] \frac{\mathrm{d}t_1}{\sqrt{t_1}}\right\} \mathrm{d}t\,,$$

which becomes after integration by parts

$$\begin{aligned}\log\left(\frac{A_0 + \Delta A_0}{A_0}\right) = &\frac{c}{4n} \cdot \left\{\sqrt{\frac{\pi}{2n}} - 2\sqrt{t} \cdot \cos(nt)\right. \\ &- \cos(2nt) \cdot \int_t^\infty \cos(nt) \frac{\mathrm{d}t}{\sqrt{t}} \\ &\left. + \left[2nt - \sin(2nt)\right] \cdot \int_t^\infty \sin(nt) \frac{\mathrm{d}t}{\sqrt{t}}\right\}\,.\end{aligned} \tag{B.12}$$

Now t is supposed to be very large: in COULOMB's experiments in fact 10 oscillations were observed, so that $nt = 10\pi$. But when t is at all large the two integrals

$$\int_t^\infty \cos(nt)\frac{\mathrm{d}t}{\sqrt{t}}\,,$$
$$\int_t^\infty \sin(nt)\frac{\mathrm{d}t}{\sqrt{t}}$$

can be expressed under the forms

$$-P\cdot\sin(nt)+Q\cdot\cos(nt)\,,$$
$$P\cdot\cos(nt)+Q\cdot\sin(nt)\,,$$

where

$$P=\frac{1}{n\sqrt{t}}-1\cdot3\cdot\frac{1}{4n^3\sqrt{t^5}}+\cdots\,,$$
$$Q=1\cdot\frac{1}{2n^2\sqrt{t^3}}-1\cdot3\cdot5\cdot\frac{1}{8n^4\sqrt{t^7}}+\cdots\,,$$

are series which are at first rapidly convergent, and which enable us to calculate the numerical values of the integrals with extreme facility. These expressions were first given by M. CAUCHY, in the case of FRESNEL's integrals, to which the integrals just written are equivalent. They may readily be obtained by integration by parts, though it is not thus that they were demonstrated by M. CAUCHY. If now the above expressions be substituted for the integrals in Equation B.12, the terms containing $\sqrt{t}$ destroy each other, and for general values of t the most important term after the first contains $\frac{1}{\sqrt{t}}$. Since however t is supposed to correspond to the end of an oscillation, so that nt is a multiple of π, the coefficient of this term vanishes, and the most important term that actually remains contains only $\frac{1}{\sqrt{t^3}}$. Hence neglecting insensible quantities we get from Equation B.12

$$\log\left(\frac{A_0+\Delta A_0}{A_0}\right)=\frac{c}{4n}\cdot\sqrt{\frac{\pi}{2n}}\,. \tag{B.13}$$

We get from Equation B.11 by performing the integrations

$$\begin{aligned}\log\left(\frac{A_0+\Delta A_0}{A}\right)&=c\cdot\sqrt{\frac{\pi}{2n}}\cdot\int_0^t\sin(nt)\cdot[\cos(nt)+\sin(nt)]\,\mathrm{d}t\\&=\frac{c}{4n}\cdot\sqrt{\frac{\pi}{2n}}\cdot\{2nt+1-\cos(2nt)-\sin(2nt)\}\,,\end{aligned}$$

which becomes since nt is a multiple of π

$$\log\left(\frac{A_0+\Delta A_0}{A}\right) = \frac{c}{4n}\cdot\sqrt{\frac{\pi}{2n}}\cdot 2nt\,. \tag{B.14}$$

We get from Equation B.13 and Equation B.14

$$2nt\cdot\log\left(\frac{A_0+\Delta A_0}{A_0}\right) = \log\left(\frac{A_0+\Delta A_0}{A}\right) = \log\left(\frac{A_0+\Delta A_0}{A_0}\right) + \log\left(\frac{A_0}{A}\right)\,,$$

whence

$$\log\left(\frac{A_0+\Delta A_0}{A_0}\right) = \frac{1}{(2nt-1)}\cdot\log\left(\frac{A_0}{A}\right)\,, \tag{B.15}$$

and the same relation exists between the common logarithms of the arcs, which are proportional to the NAPIERian logarithms. Now $\log(A_0)-\log(A)$ is the quantity immediately deduced from experiment, and $\log(A_0+\Delta A_0)-\log(A_0)$ is the correction to be applied, in consequence of the circumstance that the motion began from rest. Instead of applying the proportionate correction $+\frac{1}{(2nt-1)}$ to the difference of the logarithms, we may apply it to the deduced value of $\sqrt{\mu'}$, which is proportional to the difference of the logarithms. In COULOMB's experiments 10 oscillations were observed, and therefore $2nt = 20\pi$, and $\frac{1}{(2nt-1)} = 0.01617$, and the uncorrected value of $\sqrt{\mu'}$ being 0.0555, we get 0.0009 for the correction, giving $\sqrt{\mu'} = 0.0564$.

C. Note C. Article 50.

The results mentioned in this article were originally given without demonstration; but as the mode in which they were obtained is short, and by no means obvious, I have thought it advisable to add the demonstrations.
In order that the right-hand members of Equation 2.138 may be perfect differentials, we must have

$$\frac{\mathrm{d}\delta}{\mathrm{d}y}+\frac{\mathrm{d}\omega'''}{\mathrm{d}x}=0\,,\quad \frac{\mathrm{d}\delta}{\mathrm{d}z}+\frac{\mathrm{d}\omega'}{\mathrm{d}y}=0\,,\quad \frac{\mathrm{d}\delta}{\mathrm{d}x}+\frac{\mathrm{d}\omega''}{\mathrm{d}z}=0\,, \tag{I}$$

$$\frac{\mathrm{d}\delta}{\mathrm{d}z}-\frac{\mathrm{d}\omega''}{\mathrm{d}x}=0\,,\quad \frac{\mathrm{d}\delta}{\mathrm{d}x}-\frac{\mathrm{d}\omega'''}{\mathrm{d}y}=0\,,\quad \frac{\mathrm{d}\delta}{\mathrm{d}y}-\frac{\mathrm{d}\omega'}{\mathrm{d}z}=0\,, \tag{II}$$

$$\frac{\mathrm{d}\omega''}{\mathrm{d}y}+\frac{\mathrm{d}\omega'''}{\mathrm{d}z}=0\,,\quad \frac{\mathrm{d}\omega'''}{\mathrm{d}z}+\frac{\mathrm{d}\omega'}{\mathrm{d}x}=0\,,\quad \frac{\mathrm{d}\omega'}{\mathrm{d}x}+\frac{\mathrm{d}\omega''}{\mathrm{d}y}=0\,, \tag{III}$$

Equation III give

$$\frac{\mathrm{d}\omega'}{\mathrm{d}x}=0\,,\quad \frac{\mathrm{d}\omega''}{\mathrm{d}y}=0\,,\quad \frac{\mathrm{d}\omega'''}{\mathrm{d}z}=0\,. \tag{IV}$$

In the particular case in which $\delta = 0$, Equation I, Equation II, and Equation IV give

$$\mathrm{d}\omega'=0\,,\quad \mathrm{d}\omega''=0\,,\quad \mathrm{d}\omega'''=0\,,$$

and therefore ω', ω'', and ω''' are constant as stated in Art. 50., page 193. In the general case Equation I, Equation II, and Equation IV give for the differentials of ω', ω'', and ω''' the following expressions:

$$\left.\begin{aligned}\mathrm{d}\omega' &= -\frac{\mathrm{d}\delta}{\mathrm{d}z}\,\mathrm{d}y+\frac{\mathrm{d}\delta}{\mathrm{d}y}\,\mathrm{d}z\,,\\ \mathrm{d}\omega'' &= -\frac{\mathrm{d}\delta}{\mathrm{d}x}\,\mathrm{d}z+\frac{\mathrm{d}\delta}{\mathrm{d}z}\,\mathrm{d}x\,,\\ \mathrm{d}\omega''' &= -\frac{\mathrm{d}\delta}{\mathrm{d}y}\,\mathrm{d}x+\frac{\mathrm{d}\delta}{\mathrm{d}x}\,\mathrm{d}y\,.\end{aligned}\right\} \tag{V}$$

In order that the right-hand members of these equations may be perfect differentials, we must have

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}y\,\mathrm{d}z}=0\,,\quad \frac{\mathrm{d}^2\delta}{\mathrm{d}z\,\mathrm{d}x}=0\,,\quad \frac{\mathrm{d}^2\delta}{\mathrm{d}x\,\mathrm{d}y}=0\,, \tag{VI}$$

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}y^2}+\frac{\mathrm{d}^2\delta}{\mathrm{d}z^2}=0\,,\quad \frac{\mathrm{d}^2\delta}{\mathrm{d}z^2}+\frac{\mathrm{d}^2\delta}{\mathrm{d}x^2}=0\,,\quad \frac{\mathrm{d}^2\delta}{\mathrm{d}x^2}+\frac{\mathrm{d}^2\delta}{\mathrm{d}y^2}=0\,,$$

and therefore

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}x^2}=0\,,\quad \frac{\mathrm{d}^2\delta}{\mathrm{d}y^2}=0\,,\quad \frac{\mathrm{d}^2\delta}{\mathrm{d}z^2}=0\,. \tag{VII}$$

Equation VI and Equation VII give

$$\mathrm{d}\frac{\mathrm{d}\delta}{\mathrm{d}x}=0\,,\quad \mathrm{d}\frac{\mathrm{d}\delta}{\mathrm{d}y}=0\,,\quad \mathrm{d}\frac{\mathrm{d}\delta}{\mathrm{d}z}=0\,,$$

so that $\frac{\mathrm{d}\delta}{\mathrm{d}x}$, $\frac{\mathrm{d}\delta}{\mathrm{d}y}$ and $\frac{\mathrm{d}\delta}{\mathrm{d}z}$ are constant. Substituting in Equation V and integrating, and then substituting in Equation 2.138 the resulting expressions for ω', ω'', and ω''', and integrating again, we shall obtain the results given in Art. 50, page 193.

D. Supplement to a Memoir On Some Cases Of Fluid Motion

In a memoir which the Society did me the honour to publish in their *Transactions*[1], I shewed that when a box whose interior is of the form of a rectangular parallelepiped is filled with fluid and made to perform small oscillations the motion of the box will be the same as if the fluid were replaced by a solid having the same mass, centre of gravity, and principal axes as the solidified fluid, but different moments of inertia about those axes. The box is supposed to be closed on all sides, and it is also supposed that the box itself and the fluid within it were both at rest at the beginning of the motion. The investigation was founded upon the ordinary equations of Hydrodynamics, which depend upon the hypothesis of the absence of any tangential force exerted between two adjacent portions of a fluid in motion, a hypothesis which entails as a necessary consequence the equality of pressure in all directions. The particular case of motion under consideration appears to be of some importance, because it affords an accurate means of comparing with experiment the common theory of fluid motion, which depends upon the hypothesis just mentioned. In my former paper, I gave a series by means of which the numerical values of the principal moments of the solid which may be substituted for the fluid might be calculated with facility. The present supplement contains a different series for the same purpose, which is more easy of numerical calculation than the former. The comparison of the two series may also be of some interest in an analytical point of view, since they appear under very different forms. I have taken the present opportunity of mentioning the results of some experiments which I have performed on the oscillations of a box, such as that under consideration . The experiments were not performed with sufficient accuracy to entitle them to be described in detail.

The calculation of the motion of fluid in a rectangular box is given in the 13th article of my former paper, page 57. I shall not however in the first instance restrict myself to a rectangular parallelepiped, since the simplification which I am about to give applies more generally. Suppose then the problem to be solved to be the following. A vessel whose interior surface is composed of any cylindrical surface and of two planes perpendicular to the generating lines of the cylinder is filled with a homogeneous, incompressible fluid ; the vessel and the fluid within it having been at first at rest, the former is then moved in any manner; it is now required to

[1]Vol. VIII. Part I. p. 105., see Part I

determine the motion of the fluid at any instant, supposing that at that instant the vessel has no motion of rotation about an axis parallel to the generating lines of the cylinder.

I shall adopt the notation of my former paper. u, v, w are the resolved parts of the velocity at any point along the rectangular axes of x, y, z. Since the motion begins from rest we shall have $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ being an exact differential $\mathrm{d}\phi$. Let the rectangular axes to which the fluid is referred be fixed relatively to the vessel, and let the axis of x be parallel to the generating lines of the cylindrical surface. The instantaneous motion of the vessel may be decomposed into a motion of translation, and two motions of rotation about the axes of y and z respectively; for by hypothesis there is no motion of rotation about the axis of x. According to the principles of my former paper, the instantaneous motion of the fluid will be the same as if it had been produced directly by impact, the impact being such as to give the vessel the velocity which it has at the instant considered. We may also consider separately the motion of translation of the vessel, and each of the motions of rotation; the actual motion of the fluid will be compounded of those which correspond to each of the separate motions of the vessel. For my present purpose it will be sufficient to consider one of the motions of rotation, that which takes place round the axis of z for instance. Let ω be the angular velocity about the axis of z, ω being considered positive when the vessel turns from the axis of x to that of y. It is easy to see that the instantaneous motion of the cylindrical surface is such as not to alter the volume of the interior of the vessel, supposing the plane ends fixed, and that the same is true of the instantaneous motion of the ends. Consequently we may consider separately the motion of the fluid due to the motion of the cylindrical surface, and to that of the ends. Let ϕ_c be the part of ϕ due to the motion of the cylindrical surface, ϕ_e the part due to the motion of the ends. Then we shall have

$$\phi = \phi_c + \phi_e\,. \tag{D.1}$$

Consider now the motion corresponding to a value of ϕ, wxy. It will be observed that wxy satisfies Equation 4.36 of my former paper, which ϕ is to satisfy. Corresponding to this value of ϕ we have

$$u = \omega \cdot y\,, \quad v = \omega \cdot x\,, \quad w = 0\,.$$

Hence the velocity, corresponding to this motion, of a particle of fluid in contact with the cylindrical surface of the vessel, resolved in a direction perpendicular to the surface, is the same as the velocity of the surface itself resolved in the same direction, and therefore the fluid does not penetrate into, nor separate from the cylindrical surface. The velocity of a particle in contact with either of the plane ends, resolved in a direction perpendicular to the surface, is equal and opposite to the velocity of the surface itself resolved in the same direction. Hence we shall get

the complete value of ϕ by adding the part already found, namely wxy, to *twice* the part due to the motion of the plane ends. We have therefore,

$$\phi = \omega xy + 2\phi_e = 2\phi_e - \omega xy\,, \text{ by Equation D.1} \tag{D.2}$$

and

$$\phi_c - \phi_e = \omega xy\,. \tag{D.3}$$

Hence whenever either ϕ_c or ϕ_e can be found, the complete solution of the problem will be given by Equation D.2. And even when both these functions can be obtained independently, Equation D.2 will enable us to dispense with the use of one of them, and Equation D.3 will give a relation between them. In this case Equation D.3 will express a theorem in pure analysis, a theorem which will sometimes be very curious, since the analytical expressions for ϕ_c and ϕ_e will generally be totally different in form. The problem admits of solution in the case of a circular cylinder terminated by planes perpendicular to its axis, and in the case of a rectangular parallelepiped. In the former case, the numerical calculation of the moments of inertia of the solid by which the fluid may be replaced would probably be troublesome, in the latter it is extremely easy. I proceed to consider this case in particular.

Let the rectangular axes to which the fluid is referred coincide with three adjacent edges of the parallelepiped, and let a, b, c be the lengths of the edges. The motion which it is proposed to calculate is that which arises from a motion of rotation of the box about an axis parallel to that of z and passing through the centre of the parallelepiped. Consequently in applying Equation D.2 we must for a moment conceive the axis of z to pass through the centre of the parallelepiped, and then transfer the origin to the corner, and we must therefore write $\omega \cdot \left(x - \frac{1}{2}a\right) \cdot \left(y - \frac{1}{2}b\right)$ for wxy. In the present case the cylindrical surface consists of the four faces which are parallel to the axis of x, and the remaining faces form the plane ends. The motion of the face xy and the opposite face has evidently no effect on the fluid, so that ϕ_c will be the part of ϕ due to the motion of the face xz and the opposite face. The value of this quantity is given near the middle of page 58 in my former paper. We have then by the second of the Equation D.2

$$\begin{aligned}\phi = \frac{8\omega a^2}{\pi^3} \cdot \Sigma_0 \frac{1}{n^3} \cdot \frac{\left(e^{\frac{n\pi b}{a}} - 1\right) \cdot e^{-\frac{n\pi y}{a}} + \left(e^{-\frac{n\pi b}{a}} - 1\right) \cdot e^{\frac{n\pi y}{a}}}{\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)} \cdot \cos\left(\frac{n\pi x}{a}\right) \\ - \omega \cdot \left(x - \frac{1}{2}a\right) \cdot \left(y - \frac{1}{2}b\right)\,,\end{aligned} \tag{D.4}$$

the sign Σ_0 denoting the sum corresponding to all odd integral values of n from 1 to ∞. This value of ϕ expresses completely the motion of the fluid due to a motion of rotation of the box about an axis parallel to that of z, and passing through the centre of its interior.

Suppose now the motion to be very small, so that the square of the velocity may be neglected. Then, p denoting the part of the pressure due to the motion, we shall have $p = -\rho \cdot \frac{\mathrm{d}\phi}{\mathrm{d}t}$. Also in finding $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ we may suppose the axes to be fixed in space, since by taking account of their motion we should only introduce terms depending on the square of the velocity. In fact, if for the sake of distinction we denote the co-ordinates of a fluid particle referred to the moveable axes by x', y', while x, y denote its co-ordinates referred to axes fixed in space, which after differentiation with respect to t we may suppose to coincide with the moveable axes at the instant considered, and if we denote the differential coefficient of ϕ with respect to t by $\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)$ when x, y, t are the independent variables, and by $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ when x', y', t are the independent variables, we shall have[2]

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) = \frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\mathrm{d}\phi}{\mathrm{d}x'}\frac{\mathrm{d}x'}{\mathrm{d}t} + \frac{\mathrm{d}\phi}{\mathrm{d}y'}\frac{\mathrm{d}y'}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}t} + u \cdot \frac{\mathrm{d}x'}{\mathrm{d}t} + v \cdot \frac{\mathrm{d}y'}{\mathrm{d}t};$$

for $\frac{\mathrm{d}\phi}{\mathrm{d}x'}$, $\frac{\mathrm{d}\phi}{\mathrm{d}y'}$ mean absolutely the same as $\frac{\mathrm{d}\phi}{\mathrm{d}x}$, $\frac{\mathrm{d}\phi}{\mathrm{d}y}$, and are therefore equal to u, v respectively. Now $\frac{\mathrm{d}x'}{\mathrm{d}t}$, $\frac{\mathrm{d}y'}{\mathrm{d}t}$, depending on the motion of the axes, are small quantities of the order ω; their values are in fact ωy, $-\omega x$; so that, omitting small quantities of the order ω^2, we have

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) = \frac{\mathrm{d}\phi}{\mathrm{d}t}.$$

We shall therefore find the value of p from that of ϕ by merely writing $-\rho \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t}$ for ω. In order to determine the motion of the box it will be necessary to find the resultant of the fluid pressures on its several faces. As shewn in my former paper, these pressures will have no resultant force, but only a resultant couple, of which the axis will evidently be parallel to that of z. In calculating this couple, it is immaterial whether we take the moments about the axis of z, or about a line parallel to it passing through the centre of the parallelepiped: suppose that we adopt the latter plan. If we reckon the couple positive when it tends to turn the box from the axis of x to that of y we shall evidently have

$$-\int_0^a \int_0^c p_{y=0}\left(x - \frac{a}{2}\right)\mathrm{d}x\,\mathrm{d}z$$

for the part arising from the pressure on the face xz, and

$$\int_0^b \int_0^c p_{x=0}\left(y - \frac{b}{2}\right)\mathrm{d}y\,\mathrm{d}z$$

[2]It may be very easily proved by means of this equation, combined with the general equation which determines p, that whether the velocity be great or small the fluid will have the same effect on the motion of the box as the solid of which the moment of inertia is determined in this paper on the supposition that the motion is small.

for the part arising from the pressure on the face yz. It is easily seen from Equation D.4 that $p_{x=a} = -p_{x=0}$, and $p_{y=b} = -p_{y=0}$, so that the couples due to the pressures on the faces xz, yz are equal to the couples due to the pressures on the opposite faces respectively. In order, therefore, to find the whole couple we have only got to double the part already found. As the integrations do not present the slightest difficulty, it will be sufficient to write down the result. It will be found that the whole couple is equal to $-C \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t}$, where

$$C = \frac{\rho abc}{12} \cdot \left(b^2 - 3a^2\right) + \frac{64\rho a^4 c}{\pi^5} \cdot \Sigma_0 \frac{1}{n^5} \cdot \frac{1 - e^{-\frac{n\pi b}{a}}}{1 + e^{-\frac{n\pi b}{a}}} \tag{D.5}$$

This expression has been simplified after integration by putting for $\Sigma_0 \frac{1}{n^4}$ its value $\frac{\pi^4}{96}$.

It appears then that the effect of the inertia of the fluid is to increase the moment of inertia of the box about an axis passing through its centre and parallel to the edge c by the quantity C. In Equation 4.40 of my former paper, there is given an expression for C which is apparently very different from that given by Equation D.5, but the numerical values of the two expressions are necessarily the same. If we denote the moment of inertia of the fluid supposed to be solidified by $C_\prime$, we shall have $C_\prime = \frac{\rho abc \cdot \left(a^2+b^2\right)}{12}$; and if we put

$$\frac{a}{b} = r\,, \quad \frac{C}{C_\prime} = f(r)\,,$$

and treat Equation D.5 as Equation 4.40 of my former paper was treated, we shall find

$$f(r) = \frac{1}{(1+r^2)} \cdot \left\{ 1 - 3r^2 + 2r^3 \right. \\ \left. \cdot \left(1.260497 - 1.254821 \cdot \Sigma_0 \frac{1}{n^5} \operatorname{versin}(2\theta_n) \right) \right\} \tag{D.6}$$

where

$$\text{tab. log tan}(\theta_n) = 10 - 0.6821882 \cdot \frac{n}{r}\,.$$

Equation D.6 is true, (except as regards the decimals omitted,) whatever be the value of r; but for convenience of calculation it will be proper to take r less than 1, that is, to choose for a the smaller of the two a, b. The value of $f(r)$ given by Equation D.6 is apparently very different from that given at the bottom of page **??** of my former paper, but any one may easily satisfy himself as to equivalence of the two expressions by assigning to r a value at random, and calculating the value of $f(r)$ from the two expressions separately. The Equation D.6 is however preferable to the other, especially when we have to calculate the value of $f(r)$ for small values

of r. The infinite series contained in Equation D.6 converges with such rapidity that in the most unfavourable case, that is, when $r = 1$ nearly, the omission of all terms after the first would only introduce an error of about 0.000003 in the value of $f(r)$.

For the sake of shewing the manner in which $f(r)$ alters with r, I have calculated the following values of the function. Equation D.6 shews that $f'(r) = 0$ when $r = 0$; and $f'(r)$ is also $= 0$ when $r = 1$, since $f\left(\frac{1}{r}\right) = f(r)$.

r	$f(r)$	r	$f(r)$
0.0	1	0.6	0.3374
0.1	0.9629	0.7	0.2521
0.2	0.8655	0.8	0.1958
0.3	0.7322	0.9	0.1655
0.4	0.5873	1	0.1565
0.5	0.4512		

Table D.1.: Calculated values for $f(r)$

The experiments to which I have alluded were made with a wooden box measuring inside 8 inches by 4 square. The box weighed not quite 1 lb., and contained about $4\frac{1}{2}$ lbs. of water, so that the inertia of the water which had to be overcome was by no means small compared with that of the box. The box was suspended by two parallel threads 3 inches apart and between 4 and 5 feet long: it was twisted a little, and then left to itself, so that it oscillated about a vertical axis midway between the threads. The points of attachment of the threads were in a line drawn through the centre of the upper face parallel to one of its sides, and were equidistant from the centre. The weight of the box when empty, the length and distance of the threads, the time of oscillation, and the known length of the seconds' pendulum are data sufficient for determining the moment of inertia of the box about a vertical axis passing through its centre. When the box is filled with water the same quantities determine the moment of inertia of the box and the water it contains, whence the moment of inertia of the water alone is obtained by subtraction. It is supposed here that the centre of gravity of the box coincides with the centre of gravity of its interior volume. In the following experiments a different face of the box was uppermost each time. In Nos. 1 and 2 the long edges of the box were vertical, in Nos. 3 and 4 they were horizontal. In all cases the inertia determined by experiment was a little greater than that resulting from theory: the difference will be given in fractional parts of the latter. The difference was 1/21 in No. 1, 1/13 in No. 2, 1/17 in No. 3, and 1/21 in No. 4. On referring to the table at the end of the last chapter, it will be seen that the ratio of the moment of inertia of the fluid to what it would be if the fluid were solid is about three times as great in the last two experiments as in the first two.

I had expected beforehand to find the inertia determined by experiment a little greater than that given by theory, for this reason. In the theory, it is supposed that both the fluid itself and the surface of the box are perfectly smooth. This however is not strictly true. The box by its roughness exerts a tangential force on the fluid immediately in contact with it, and this force produces an effect on the fluid at a small distance from the surface of the box, in consequence of the internal friction of the fluid itself. We may conceive the effect of this force on the time of oscillation in a general way by supposing a thin film of fluid close to the surface of the box to be dragged along with it. Consequently, the moment of inertia determined by experiment will be a little greater than it would have been had the fluid and the surface of the box been perfectly smooth.

These experiments are sufficient to shew that in the case of a vessel of about the size and shape of the one I used, filled with water, and performing small oscillations of the duration of about one second (as was the case in my experiments), the time of oscillation is not much increased by friction; at least, if we suppose, as there is reason for supposing, that the effect of friction does not depend on the nature of the surface of the box. They are not however sufficiently exact to allow us to place any reliance on the accuracy of the small differences between the results of experiment, and of the common theory of fluid motion, and consequently they are useless as tests of any theory of friction.

E. Exact Differential. Description of Mathematical Properties

In multivariate calculus, i.e. the extension of calculus in one variable to calculus with functions of several variables, a differential or differential form is said to be *exact* or perfect (*exact differential*), as contrasted with an inexact differential, if it is equal to the general differential $\mathrm{d}\phi$ for some differentiable function ϕ in an orthogonal coordinate system (hence ϕ is a multivariable function whose variables are independent, as they are always expected to be when treated in multivariable calculus).
Despite the example is shewn in three dimensions here, the definitions of exact differentials for other dimensions are structurally similar to the three dimensional definition. In three dimensions, a form of the type

$$u(x, y, z)\ \mathrm{d}x + v(x, y, z)\ \mathrm{d}y + w(x, y, z)\ \mathrm{d}z \tag{E.1}$$

is called a differential form. This form is called exact on an open domain $\mathbb{R}^3$ in space if there exists some differentiable scalar function $\phi = \phi(x, y, z)$ defined on D such that

$$\mathrm{d}\phi \equiv \left(\frac{\partial Q}{\partial x}\right)_{y,z} \mathrm{d}x + \left(\frac{\partial Q}{\partial y}\right)_{x,z} \mathrm{d}y + \left(\frac{\partial Q}{\partial z}\right)_{x,y} \mathrm{d}z, \tag{E.2}$$

$$\mathrm{d}\phi = u\ \mathrm{d}x + v\ \mathrm{d}y + w\ \mathrm{d}z$$

throughout D, where x, y, z are orthogonal co-ordinates (e.g., Cartesian, cylindrical, or spherical co-ordinates). In other words, in some open domain of a space, a differential form is an exact differential if it is equal to the general differential of a differentiable function in an orthogonal coordinate system.

F. Napierian Logarithm

The first definition of the logarithm was constructed by Napier and popularized through his posthumous pamphlet (Napier 1619). In this pamphlet, Napier sought to reduce the operations of multiplication, division, and root extraction to addition and subtraction. To this end, he defined the "logarithm" L of a number N by

$$N = 10^7 \cdot \left(1 - \frac{1}{10^7}\right)^L , \tag{F.1}$$

written $\mathrm{NapLog}(N) = L$ or $\mathfrak{l}(N) = L$.
This definition leads to the remarkable relations

$$\begin{aligned}
\sqrt{N_1 N_2} &= 10^7 \cdot (1 - 10^{-7})^{(L_1+L_2)/2} \\
10^{-7} \cdot N_1 N_2 &= 10^7 \cdot (1 - 10^{-7})^{L_1+L_2} \\
10^7 \cdot \frac{N_1}{N_2} &= 10^7 \cdot (1 - 10^{-7})^{L_1-L_2}
\end{aligned}$$

which give the identities

$$\begin{aligned}
\mathfrak{l}(\sqrt{N_1 N_2}) &= \frac{1}{2} \cdot (\mathfrak{l}(N_1) + \mathfrak{l}(N_2)) \\
\mathfrak{l}(10^{-7} \cdot N_1 N_2) &= \mathfrak{l}(N_1) + \mathfrak{l}(N_2) \\
\mathfrak{l}\left(10^7 \cdot \frac{N_1}{N_2}\right) &= \mathfrak{l}(N_1) - \mathfrak{l}(N_2)
\end{aligned}$$

While Napier's definition is different from the modern one (in particular, it decreases with increasing N, but also fails to satisfy a number of properties of the modern logarithm), it provides the desired property of transforming multiplication into addition. The Napierian logarithm can be given in terms of the modern logarithm by solving Equation F.1 for L, giving

$$\mathfrak{l}(N) = \frac{\log\left(\frac{10^7}{N}\right)}{\log\left(\frac{10^7}{10^7-1}\right)} , \tag{F.2}$$

or

$$\mathfrak{l}(N) \approx 23025851(7 - \log_{10} N) ,$$

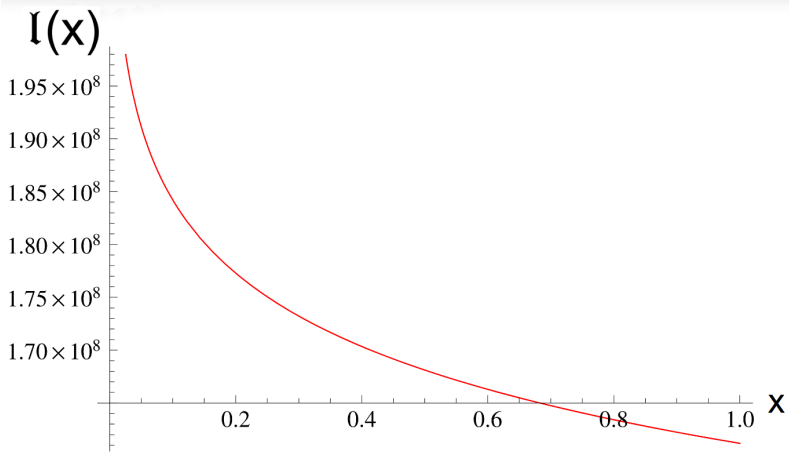

Figure F.1.: A plot of the NAPIERian logarithm $ɭ(x)$

where $23025851 \approx 10^7 \ln(10)$.
Because a ratio of logarithms appears in this expression, any logarithm base b can be used as long as the same value of b is used for both numerator and denominator. NAPIERian logarithms are essentially natural logarithms with decimal points shifted 7 places rightward and with sign reversed. For instance the logarithmic values

$$\ln(.5000000) = -0.6931471806$$
$$\ln(.3333333) = -1.0986123887$$

would have the corresponding NAPIERian logarithms:

$$ɭ(5000000) = 6931472$$
$$ɭ(3333333) = 10986124\,.$$

G. Sir George Gabriel Stokes

Sir George Gabriel Stokes, 1st Baronet, FRS (13 August 1819 – 1 February 1903) was an Irish physicist and mathematician. Born in County Sligo, Ireland, Stokes spent all of his career at the University of Cambridge, where he was the Lucasian Professor of Mathematics from 1849 until his death in 1903. As a physicist, Stokes made seminal contributions to fluid mechanics, including the Navier–Stokes equations; and to physical optics, with notable works on polarization and fluorescence. As a mathematician, he popularised "Stokes' theorem" in vector calculus and contributed to the theory of asymptotic expansions. Stokes, along with Felix Hoppe-Seyler, first demonstrated the oxygen transport function of haemoglobin, and shewed colour changes produced by the aeration of haemoglobin solutions.

Stokes was made a baronet by the British monarch in 1889. In 1893 he received the Royal Society's Copley Medal, then the most prestigious scientific prize in the world, "for his researches and discoveries in physical science". He represented Cambridge University in the British House of Commons from 1887 to 1892, sitting as a Conservative. Stokes also served as president of the Royal Society from 1885 to 1890 and was briefly the Master of Pembroke College, Cambridge.